Modeling and simulation of compressible two-phase flows

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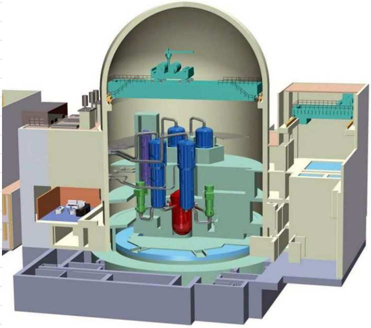
Yannick Meyapin: Master student

Industrial examples of two-phase flows

- Pressurized Water Reactors
- Liquefied Natural Gas (LNG) carriers

Typical Pressurized Water Reactor



Source: U.S. Nuclear Regulatory Commission

Two-phase flows in nature

Some typical examples

- Wave breaking phenomena
- Powder-snow avalanches

Derivation of two-phase models

Great lines of derivation procedure (Ishii, 1975; Rovarch, 2006)

- Consider governing equations in each phase (+ interface conditions):

$$\frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\vec{u}) = \nabla\cdot\sigma + \rho S, \quad \psi \in \{1, \vec{u}, e + \frac{1}{2}|\vec{u}|^2\}$$

- Introduce the characteristic function of each phase:

$$\chi_k(\vec{x}, t) = \begin{cases} 1, & \vec{x} \in \Omega_k(t) \\ 0, & \text{otherwise} \end{cases} \quad : \quad \frac{\partial\chi_k}{\partial t} + \vec{u}_i\cdot\nabla\chi_k = 0, \; k \in \{+, -\}$$

- Combination of two equations:

$$\frac{\partial(\chi_k\rho\psi)}{\partial t} + \nabla\cdot(\chi_k\rho\psi\vec{u}) = \nabla\cdot(\chi_k\sigma) + \big(\sigma - \rho\psi(\vec{u} - \vec{u}_i)\big)\cdot\vec{n}_k\delta_k + \chi_k\rho S$$

Derivation of the six-equations model

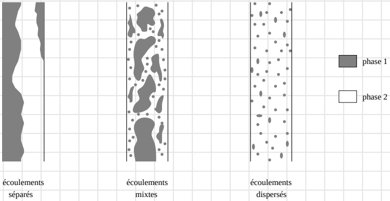
Averaging of governing equations (Ishii, 1975)

Reynolds axioms:

Linearity: $\overline{\lambda f + \mu g} = \lambda \overline{f} + \mu \overline{g}$

Idempotency: $\overline{\overline{f} g} = \overline{f}\ \overline{g}$

Commutativity: $\overline{\frac{\partial f}{\partial t}} = \frac{\partial \overline{f}}{\partial t}, \quad \overline{\frac{\partial g}{\partial x}} = \frac{\partial \overline{g}}{\partial x}$



Volume fraction definition:

$$\alpha^{\pm} := \overline{\chi_k}, \qquad \alpha^{+}(\vec{x}, t) + \alpha^{-}(\vec{x}, t) = 1$$

- Apply this average operator to governing equations:

$$\frac{\partial \overline{(\chi_k \rho \psi)}}{\partial t} + \nabla \cdot \overline{(\chi_k \rho \psi \vec{u})} = \nabla \cdot (\overline{\chi_k \sigma}) + \overline{\left(\sigma - \rho\psi(\vec{u} - \vec{u}_i)\right) \cdot \vec{n}_k \delta_k} + \overline{\chi_k \rho S}$$

Six-equations model (in simplified form)

Governing equations are to be completed by EOS

- Mass conservation:

$$\frac{\partial(\alpha^{\pm}\rho^{\pm})}{\partial t} + \nabla\cdot(\alpha^{\pm}\rho^{\pm}\vec{u}^{\pm}) = \Gamma^{\pm}$$

- Momentum conservation:

$$\frac{\partial(\alpha^{\pm}\rho^{\pm}\vec{u}^{\pm})}{\partial t} + \nabla\cdot(\alpha^{\pm}\rho^{\pm}\vec{u}^{\pm}\otimes\vec{u}^{\pm}) + \alpha^{\pm}\nabla p = \alpha^{\pm}\rho^{\pm}\vec{g} + \nabla\cdot(\alpha^{\pm}\tau^{\pm}) + M^{\pm}$$

- Energy conservation

$$\frac{\partial(\alpha^{\pm}\rho^{\pm}E^{\pm})}{\partial t} + \nabla\cdot(\alpha^{\pm}\rho^{\pm}H^{\pm}\vec{u}^{\pm}) + p\frac{\partial\alpha^{\pm}}{\partial t} = \alpha^{\pm}\rho^{\pm}\vec{g}\cdot\vec{u}^{\pm} + \nabla\cdot(\alpha^{\pm}\tau^{\pm}\vec{u}^{\pm}) - \nabla\cdot(\alpha^{\pm}\vec{q}^{\pm}) + Q^{\pm}$$

Reduce the number of variables

Relaxation process modeling

Method:

- Add relaxation terms into momentum and energy conservation equations:

$$\vec{F}_d := \frac{\kappa}{\varepsilon} \frac{\alpha^+\rho^+\alpha^-\rho^-}{\alpha^+\rho^+ + \alpha^-\rho^-}(\vec{u}^+ - \vec{u}^-), \quad E_d := \vec{F}_d \cdot \bar{u}$$

- Take singular limit when $\varepsilon \to 0$ using Chapman-Enskog expansion:

$$V_\varepsilon = V + \varepsilon W + \mathcal{O}(\varepsilon^2)$$

Reference:

Y. Meyapin, D. Dutykh & M. Gisclon. *Velocity and energy relaxation in two-phase flows*. Stud. Appl. Math., **125**(2), 179–212, 2010

Four-equations model

Single velocity two-phase model

After computations we get the following system:

$$\partial_t(\alpha^\pm\rho^\pm) + \nabla\cdot(\alpha^\pm\rho^\pm\vec{u}) = 0,$$
$$\partial_t(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla p = \rho\vec{g} + \nabla\cdot\tau,$$
$$\partial_t(\rho E) + \nabla\cdot(\rho H\vec{u}) = \rho\vec{g}\cdot\vec{u} + \nabla\cdot(\tau\vec{u}) - \nabla\cdot\vec{q}.$$

$$p = p^\pm(\rho^\pm, e^\pm), \quad T = T^\pm(\rho^\pm, e^\pm)$$

$$\rho := \alpha^+\rho^+ + \alpha^-\rho^-, \quad \rho e := \alpha^+\rho^+e^+ + \alpha^-\rho^-e^-$$

$$\tau := \alpha^+\tau^+ + \alpha^-\tau^-, \quad H := E + \frac{p}{\rho}$$

This model was proposed earlier in ad-hoc way:

F. Dias, D. Dutykh, J.-M. Ghidaglia. *A two-fluid model for violent aerated flows*. Computers & Fluids, **39**(2), 283–293, 2010

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$$\partial_t(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \frac{1}{\mathrm{Ma}^2}\nabla p = \rho\vec{g} + \nabla\cdot\tau,$$
$$\partial_t(\rho E) + \nabla\cdot(\rho H\vec{u}) = \rho\vec{g}\cdot\vec{u} + \nabla\cdot(\tau\vec{u}) - \nabla\cdot\vec{q}.$$

$$p = p^\pm(\rho^\pm, e^\pm), \quad T = T^\pm(\rho^\pm, e^\pm)$$

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Two-fluid incompressible Navier-Stokes equations

Low Mach number limit $\mathrm{Ma} \to 0$ of the four-equations model

- Mass conservation equation:

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$$

- Incompressibility condition:

$$\nabla \cdot \vec{u} = 0$$

- Momentum conservation:

$$\partial_t(\rho\vec{u}) + \nabla \cdot (\rho\vec{u} \otimes \vec{u}) + \nabla p = \nabla \cdot \big(2\mu\mathbb{D}(\vec{u})\big) + \rho\vec{g}$$

$$\text{where} \quad \rho := \alpha^+\rho_0^+ + \alpha^-\rho_0^-, \quad \mu := \alpha^+\rho_0^+\nu^+ + \alpha^-\rho_0^-\nu^-$$

Reference:

Y. Meyapin, D. Dutykh & M. Gisclon. *Velocity and energy relaxation in two-phase flows*. Stud. Appl. Math., **125**(2), 179–212, 2010

Two-fluid Navier-Stokes equations with mixing

In view of applications to powder-snow avalanches

Interfaces are simply advected:

$$\begin{cases} \rho = \alpha^+ \rho_0^+ + \alpha^- \rho_0^- \\ \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0 \\ \nabla \cdot \vec{u} = 0 \end{cases} \quad \Longrightarrow \quad \partial_t \alpha^\pm + \vec{u} \cdot \nabla \alpha^\pm = 0$$

Diffuse interfaces with Fick's law (A. Majda, 1984):

$$\begin{aligned} \nabla \cdot \vec{u} &= -\nabla \cdot (\kappa \nabla \log \rho) \\ \partial_t \rho + \nabla \cdot (\rho \vec{u}) &= 0 \\ \partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p &= \nabla \cdot \big(2\mu \mathbb{D}(\vec{u})\big) + \rho \vec{g} \end{aligned}$$

Fluid volume velocity (after H. Brenner, 2006):

Change of variables: $\vec{v} := \vec{u} + \kappa \nabla \log \rho \quad \Rightarrow \quad \nabla \cdot \vec{v} = 0$

Governing equations:

$$\begin{aligned} \nabla \cdot \vec{v} &= 0 \\ \partial_t \rho + \vec{v} \cdot \nabla \rho &= \kappa \Delta \rho \\ \rho \partial_t \vec{v} + \rho(\vec{v} \cdot \nabla)\vec{v} + \nabla \pi + \kappa\, {}^t\nabla \vec{v} \nabla \rho - \kappa \nabla \rho \nabla \vec{v} &= \rho \vec{g} + \nabla \cdot \big(2\mu \mathbb{D}(\vec{v})\big) \end{aligned}$$

This model is energetically consistent:

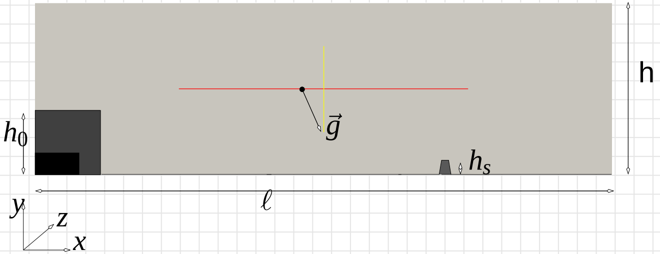
$$\partial_t \int_\Omega \rho \frac{|\vec{v}|^2}{2}\, d\vec{x} = \int_\Omega \rho \vec{g} \cdot \vec{v}\, d\vec{x} - \int_\Omega 2\mu |\mathbb{D}(\vec{v})|^2\, d\vec{x} - \int_\Omega \kappa \rho |\mathbb{A}(\vec{v})|^2\, d\vec{x}.$$

Reference:

D. Dutykh, C. Acary-Robert, D. Bresch. *Mathematical modeling of powder-snow avalanche flows*. Submitted, 2010
`http://hal.archives-ouvertes.fr/hal-00354000/`

Sliding mass interaction with obstacle

Some numerical illustrations



parameter	value
slope, θ	32°
heavy fluid density, ρ^+, kg/m^3	20
light fluid density, ρ^-, kg/m^3	1
heavy fluid kinematic viscosity, ν^+, m/s^2	4.8×10^{-4}
light fluid kinematic viscosity, ν^-, m/s^2	1.0×10^{-4}

Sliding mass interaction with obstacle

Snapshots of the volume fraction evolution

Figure: Avalanche at $t = 10\ s$. The color scale ranges from 0 to the maximum value 1.0.

Reference:

D. Dutykh, C. Acary-Robert, D. Bresch. *Mathematical modeling of powder-snow avalanche flows*. Submitted, 2010
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Sliding mass interaction with obstacle

Snapshots of the volume fraction evolution

Figure: Avalanche at $t = 25\ s$. The color scale ranges from 0 to the maximum value 0.984.

Reference:

D. Dutykh, C. Acary-Robert, D. Bresch. *Mathematical modeling of powder-snow avalanche flows*. Submitted, 2010
`http://hal.archives-ouvertes.fr/hal-00354000/`

Sliding mass interaction with obstacle

Snapshots of the volume fraction evolution

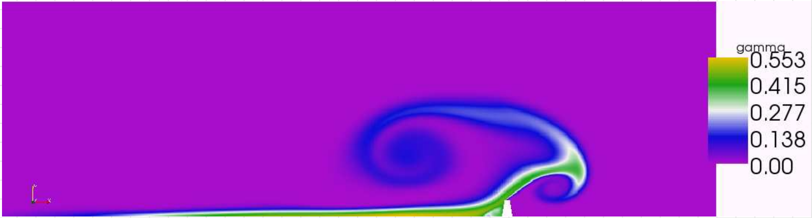

Figure: Avalanche at $t = 60\ s$. The color scale ranges from 0 to the maximum value 0.553.

Reference:

D. Dutykh, C. Acary-Robert, D. Bresch. *Mathematical modeling of powder-snow avalanche flows*. Submitted, 2010
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Conclusions & Perspectives

Conclusions:

- We presented a family of two-phase models
- Connections between different models were shown
- Incompressible two-fluid model was proposed for highly inhomogeneous two-fluid flows (powder-snow avalanches)

Perspectives:

- All-Mach solvers for single velocity two-phase flows
- Realistic boundary conditions on the soil
- Comparison with experimental data

Thank you for your attention!

http://www.lama.univ-savoie.fr/~dutykh