Relaxed variational principle for water wave modeling

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Outline

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2. Deep water example
   - gKG system
3. Shallow water examples
   - Serre system
   - SV++ system

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Luke's variational principles

J.C. Luke, JFM (1967) [Luk67]

First improvement of the classical Lagrangian: $\mathscr{L} := K - \Pi$

$$\mathcal{L} = \int_{t_1}^{t_2}\int_{\Omega} \rho\mathscr{L}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \qquad \mathscr{L} := \int_{-d}^{\eta}\Big(\phi_t + \frac{1}{2}|\boldsymbol{\nabla}_{\boldsymbol{x},y}\phi|^2 + gy\Big)\,\mathrm{d}y$$

$$\begin{aligned}
\delta\phi\colon\ & \Delta\phi = 0, \quad (\boldsymbol{x}, y) \in \Omega\times[-d, \eta], \\
\delta\phi|_{y=-d}\colon\ & \frac{\partial\phi}{\partial y} + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}d + d_t = 0, \quad y = -d, \\
\delta\phi|_{y=\eta}\colon\ & \frac{\partial\eta}{\partial t} + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\eta - \frac{\partial\phi}{\partial y} = 0, \quad y = \eta(\boldsymbol{x}, t), \\
\delta\eta\colon\ & \frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g\eta = 0, \ \ y = \eta(\boldsymbol{x}, t).
\end{aligned}$$

- We obtain the water wave problem by varying η and ϕ

Generalization of the Lagrangian density

D. Clamond & D. Dutykh (2012), [CD12]

$\tilde{\phi} := \phi(\mathbf{x}, y = \eta(\mathbf{x}, t), t)$: quantity at the free surface

$\check{\phi} := \phi(\mathbf{x}, y = -d(\mathbf{x}, t), t)$: value at the bottom

Equivalent form of Luke's lagrangian:

$$\mathscr{L} = \tilde{\phi}\eta_t + \check{\phi}d_t - \frac{1}{2}g\eta^2 + \frac{1}{2}gd^2 - \int_{-d}^{\eta} \left[\frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{1}{2}\phi_y^2\right] \mathrm{d}y$$

Explicitly introduce the velocity field: $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, $v = \phi_y$

$$\mathscr{L} = \tilde{\phi}\eta_t + \check{\phi}d_t - \frac{1}{2}g\eta^2 - \int_{-d}^{\eta} \left[\frac{1}{2}(\boldsymbol{u}^2 + v^2) + \boldsymbol{\mu}\cdot(\boldsymbol{\nabla}\phi - \boldsymbol{u}) + \nu(\phi_y - v)\right] \mathrm{d}y$$

$\boldsymbol{\mu}, \nu$: Lagrange multipliers or pseudo-velocity field

Generalization of the Lagrangian density

D. Clamond & D. Dutykh (2012), [CD12]

Relaxed variational principle:

$$\mathscr{L} = (\eta_t + \tilde{\boldsymbol{\mu}} \cdot \boldsymbol{\nabla}\eta - \tilde{\nu})\tilde{\phi} + (d_t + \check{\boldsymbol{\mu}} \cdot \boldsymbol{\nabla} d + \check{\nu})\check{\phi} - \frac{1}{2} g\eta^2 + \int_{-d}^{\eta} \Big[\boldsymbol{\mu} \cdot \boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^2 + \nu v - \frac{1}{2} v^2 + (\boldsymbol{\nabla} \cdot \boldsymbol{\mu} + \nu_y)\phi\Big] \, \mathrm{d}y$$

Classical formulation (for comparison):

$$\mathscr{L} = \tilde{\phi}\eta_t + \check{\phi} d_t - \frac{1}{2} g\eta^2 - \int_{-d}^{\eta} \Big[\frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{1}{2}\phi_y^2\Big] \, \mathrm{d}y$$

Degrees of freedom: $\eta, \phi; \boldsymbol{u}, v; \boldsymbol{\mu}, \nu$

Outline

Deep water approximation

Choice of the ansatz:

$$\{\phi; \boldsymbol{u}; v; \mu; \nu\} \approx \{\tilde{\phi}; \tilde{u}; \tilde{v}; \tilde{\boldsymbol{\mu}}; \tilde{\nu}\} e^{\kappa(y-\eta)}$$

$$2\kappa\mathscr{L} = 2\kappa\tilde{\phi}\eta_t - g\kappa\eta^2 + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2 - \tilde{u}\cdot(\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}\boldsymbol{\nabla}\eta) - \kappa\tilde{v}\tilde{\phi}$$

- generalized Klein-Gordon equations:

$$\begin{aligned} \eta_t + \frac{1}{2}\kappa^{-1}\boldsymbol{\nabla}^2\tilde{\phi} - \frac{1}{2}\kappa\tilde{\phi} &= \frac{1}{2}\tilde{\phi}[\boldsymbol{\nabla}^2\eta + \kappa(\boldsymbol{\nabla}\eta)^2] \\ \tilde{\phi}_t + g\eta &= -\frac{1}{2}\boldsymbol{\nabla}\cdot\left[\tilde{\phi}\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}^2\boldsymbol{\nabla}\eta\right] \end{aligned}$$

$$\mathcal{H} = \int_\Omega \left\{\frac{1}{2}g\eta^2 + \frac{1}{4}\kappa^{-1}[\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}\boldsymbol{\nabla}\eta]^2 + \frac{1}{4}\kappa\tilde{\phi}^2\right\} \mathrm{d}\boldsymbol{x}$$

Deep water approximation

Choice of the ansatz:

$$\{\phi; \boldsymbol{u}; v; \mu; \nu\} \approx \{\tilde{\phi}; \tilde{u}; \tilde{v}\,; \tilde{\boldsymbol{\mu}}; \tilde{\nu}\} e^{\kappa(y-\eta)}$$

$$2\kappa\mathscr{L} = 2\kappa\tilde{\phi}\eta_t - g\kappa\eta^2 + \frac{1}{2}\tilde{u}^{\,2} + \frac{1}{2}\tilde{v}^{\,2} - \tilde{u}\cdot(\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}\boldsymbol{\nabla}\eta) - \kappa\tilde{v}\tilde{\phi}$$

- generalized Klein-Gordon equations:

$$\begin{aligned} \eta_t + \frac{1}{2}\kappa^{-1}\boldsymbol{\nabla}^2\tilde{\phi} - \frac{1}{2}\kappa\tilde{\phi} &= \frac{1}{2}\tilde{\phi}[\boldsymbol{\nabla}^2\eta + \kappa(\boldsymbol{\nabla}\eta)^2] \\ \tilde{\phi}_t + g\eta &= -\frac{1}{2}\boldsymbol{\nabla}\cdot\left[\tilde{\phi}\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}^2\boldsymbol{\nabla}\eta\right] \end{aligned}$$

$$\mathbb{M}\boldsymbol{z}_t + \mathbb{K}\boldsymbol{z}_x + \mathbb{L}\boldsymbol{z}_y = \boldsymbol{\nabla}_{\boldsymbol{z}}\mathcal{S}(\boldsymbol{z})$$

Comparison with exact Stokes wave

- Cubic Zakharov Equations (CZE):

$$\begin{aligned}\eta_t - \mathfrak{d}\tilde{\phi} &= -\boldsymbol{\nabla}\cdot(\eta\boldsymbol{\nabla}\tilde{\phi}) - \mathfrak{d}(\eta\mathfrak{d}\tilde{\phi}) + \\ &\quad \tfrac{1}{2}\boldsymbol{\nabla}^2(\eta^2\mathfrak{d}\tilde{\phi}) + \mathfrak{d}(\eta\mathfrak{d}(\eta\mathfrak{d}\tilde{\phi})) + \tfrac{1}{2}\mathfrak{d}(\eta^2\boldsymbol{\nabla}^2\tilde{\phi}), \\ \tilde{\phi}_t + g\eta &= \tfrac{1}{2}(\mathfrak{d}\tilde{\phi})^2 - \tfrac{1}{2}(\boldsymbol{\nabla}\tilde{\phi})^2 - (\eta\mathfrak{d}\tilde{\phi})\boldsymbol{\nabla}^2\tilde{\phi} - (\mathfrak{d}\tilde{\phi})\mathfrak{d}(\eta\mathfrak{d}\tilde{\phi}).\end{aligned}$$

- Phase speed :

Exact: $g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^4 + \frac{707}{384}\alpha^6 + O(\alpha^8)$

CZE: $g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \frac{1}{2}\alpha^2 + \mathbf{\frac{41}{64}}\alpha^4 + \frac{\mathbf{913}}{384}\alpha^6 + O(\alpha^8)$

gKG: $g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^4 + \frac{\mathbf{899}}{384}\alpha^6 + O(\alpha^8)$

- n-th Fourier coefficient to the leading order: $\frac{n^{n-2}\alpha^n}{2^{n-1}(n-1)!}$ (the same in gKG & Stokes but not in CZE)

Outline

Shallow water regime

Choice of a simple ansatz in shallow water

Ansatz:

$$\boldsymbol{u}(\boldsymbol{x},y,t) \approx \bar{u}(\boldsymbol{x},t), v(\boldsymbol{x},y,t) \approx (y+d)(\eta+d)^{-1}\,\tilde{v}(\boldsymbol{x},t)$$

$$\phi(\boldsymbol{x},y,t) \approx \bar{\phi}(\boldsymbol{x},t), \nu(\boldsymbol{x},y,t) \approx (y+d)(\eta+d)^{-1}\,\tilde{\nu}(\boldsymbol{x},t)$$

Lagrangian density:

$$\mathscr{L} = \bar{\phi}\eta_t - \frac{1}{2}g\eta^2 + (\eta+d)\left[\bar{\mu}\cdot\bar{u} - \frac{1}{2}\bar{u}^{\,2} + \frac{1}{3}\tilde{\nu}\tilde{v} - \frac{1}{6}\tilde{v}^{\,2} - \bar{\mu}\cdot\boldsymbol{\nabla}\bar{\phi}\right]$$

Nonlinear Shallow Water Equations:

$$h_t + \boldsymbol{\nabla}\cdot[h\bar{u}] = 0,$$

$$\bar{u}_t + (\bar{u}\cdot\boldsymbol{\nabla})\bar{u} + g\boldsymbol{\nabla}h = 0.$$

Constraining with free surface impermeability

Constraint:

$$\tilde{\nu} = \eta_t + \bar{\mu} \cdot \boldsymbol{\nabla} \eta$$

- Generalized Serre equations [Ser53]:

$$h_t + \boldsymbol{\nabla} \cdot [h\bar{u}] = 0,$$

$$\bar{u}_t + \bar{u} \cdot \boldsymbol{\nabla} \bar{u} + g\boldsymbol{\nabla} h + \frac{1}{3} h^{-1} \boldsymbol{\nabla} [h^2 \tilde{\gamma}] = (\bar{u} \cdot \boldsymbol{\nabla} h) \boldsymbol{\nabla} (h \boldsymbol{\nabla} \cdot \bar{u}) - [\bar{u} \cdot \boldsymbol{\nabla} (h \boldsymbol{\nabla} \cdot \bar{u})] \boldsymbol{\nabla} h$$

$$\tilde{\gamma} = \tilde{v}_t + \bar{u} \cdot \boldsymbol{\nabla} \tilde{v} = h\big((\boldsymbol{\nabla} \cdot \bar{u})^2 - \boldsymbol{\nabla} \cdot \bar{u}_t - \bar{u} \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \bar{u})\big)$$

This model cannot be obtained from Luke's Lagrangian:

$\delta\bar{\mu}$: $\bar{u} = \boldsymbol{\nabla} \bar{\phi} - \frac{1}{3} \tilde{v} \boldsymbol{\nabla} \eta \neq \boldsymbol{\nabla} \bar{\phi}$

Shallow water theories for arbitrary slopes

Quest for improved Saint-Venant or Savage-Hutter equations

- Dispersive nonhydrostatic extensions
- Hydrostatic and nonhydrostatic nondispersive generalizations

Existing literature:

R.F. Dressler: JHR (1978), [Dre78]

F. Bouchut *et al*: CRAS (2003), [BMCPV03]

J.B. Keller: JFM (2003), [Kel03]

Model by Bouchut & Keller (2003):

$$\left(h - \tfrac{1}{2}h^2 d_x\right)_t + \left(\frac{\log(1 - h d_x)}{-d_x} u\right)_x = 0$$

$$u_t + \left(\frac{1}{(1 - g d_x)^2}\frac{u^2}{2} + g\eta\right)_x = 0$$

Improved Saint-Venant (SV++) equations

Derivation from the relaxed Lagrangian [DC11]

Choice of the ansatz:

$$\phi \approx \bar{\phi}(\boldsymbol{x},t), \quad \boldsymbol{u} = \boldsymbol{\mu} \approx \bar{\boldsymbol{u}}(\boldsymbol{x},t), \quad v = \nu \approx \check{v}(\boldsymbol{x},t) = -d_t - \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} d$$

Lagrangian:

$$\mathscr{L} = (h_t + \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} h + h \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}})\, \bar{\phi} - \tfrac{1}{2} g \eta^2 + \tfrac{1}{2} h (\bar{\boldsymbol{u}}^2 + \check{v}^2)$$

$$h_t + \boldsymbol{\nabla} \cdot [h \bar{\boldsymbol{u}}] = 0,$$

$$[\bar{\boldsymbol{u}} - \check{v} \boldsymbol{\nabla} d]_t + \boldsymbol{\nabla} [g \eta + \tfrac{1}{2} \bar{\boldsymbol{u}}^2 + \tfrac{1}{2} \check{v}^2 + \check{v} d_t] = 0.$$

- Momentum conservation equation ($\gamma := \frac{\mathrm{d}\check{v}}{\mathrm{d}t} = \check{v}_t + (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\check{v}$):

$$[h \bar{\boldsymbol{u}}]_t + \boldsymbol{\nabla} [h \bar{\boldsymbol{u}}^2 + \tfrac{1}{2} g h^2] = (g + \gamma) h \boldsymbol{\nabla} d + h\bar{\boldsymbol{u}} \wedge \boldsymbol{\nabla} \check{v} \wedge \boldsymbol{\nabla} d$$

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Lagrangian:

$$\mathscr{L} = (h_t + \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} h + h \, \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}) \, \bar{\phi} - \tfrac{1}{2} g \, \eta^2 + \tfrac{1}{2} h \, (\bar{\boldsymbol{u}}^{\,2} + \check{v}^{\,2})$$

$$h_t + \boldsymbol{\nabla} \cdot [\, h \, \bar{\boldsymbol{u}} \,] = 0,$$

$$[\, \bar{\boldsymbol{u}} - \check{v} \, \boldsymbol{\nabla} d \,]_t + \boldsymbol{\nabla} [\, g \, \eta + \tfrac{1}{2} \bar{\boldsymbol{u}}^{\,2} + \tfrac{1}{2} \check{v}^{\,2} + \check{v} \, d_t \,] = 0.$$

- Gravity wave propagation speed in SV and SV++:

$$c_{sv} := \sqrt{gh}, \qquad c_{sv++} := \frac{\sqrt{gh}}{\sqrt{1 + |\boldsymbol{\nabla} d|^2}}$$

Structural properties of SV++

Hamiltonian structure and energy conservation [DC11]

Hamiltonian form:

$$\frac{\partial h}{\partial t} = \frac{\delta \mathscr{H}}{\delta \bar{\phi}}, \qquad \frac{\partial \bar{\phi}}{\partial t} = -\frac{\delta \mathscr{H}}{\delta h}$$

$$\mathscr{H} = \tfrac{1}{2} \int \left\{ g(h-d)^2 + h|\boldsymbol{\nabla}\bar{\phi}|^2 - \frac{h[d_t + \boldsymbol{\nabla}\bar{\phi}\cdot\boldsymbol{\nabla}d]^2}{1+|\boldsymbol{\nabla}d|^2} \right\} \mathrm{d}^2\boldsymbol{x}$$

Or equivalently:

$$\mathscr{H} = \tfrac{1}{2} \int \left\{ g\eta^2 + h\bar{\boldsymbol{u}}^{\,2} + h(\check{v} + d_t)^2 - hd_t^{\,2} \right\} \mathrm{d}^2\boldsymbol{x},$$

- Energy conservation identity:

$$\left[h\frac{|\bar{\boldsymbol{u}}|^2 + \check{v}^{\,2}}{2} + g\frac{\eta^2 - d^{\,2}}{2} \right]_t + \boldsymbol{\nabla}\cdot\left[\left(\frac{|\bar{\boldsymbol{u}}|^2 + \check{v}^{\,2}}{2} + g\eta \right) h\bar{\boldsymbol{u}} \right] = -(g+\gamma)hd_t$$

Thank you for your attention!

http://www.lama.univ-savoie.fr/~dutykh/

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