A generalized variational principle for water wave modeling

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Acknowledgements

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Outline

1. Water wave problem
2. Variational formulations
   - Lagrangian principles
   - Hamiltonian formulation
3. Generalized Lagrangian
4. Applications
   - Models in shallow water
   - Models in deep water
   - Arbitrary depth

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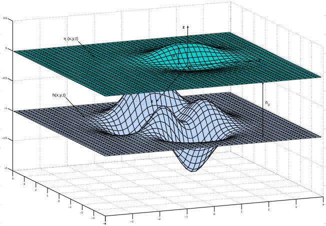
Water wave problem

Physical assumptions:

- Fluid is ideal
- Flow is incompressible
- …and irrotational, i.e. $\vec{u} = \nabla\varphi$
- Free surface is a graph
- Above free surface there is void
- Atmospheric pressure is constant

Surface tension can be also taken into account



Water wave problem - I

Mathematical formulation

- Continuity equation

$$\Delta\varphi = 0, \quad (\vec{x}, y) \in \Omega \times [-d, \eta],$$

- Kinematic bottom condition

$$\frac{\partial\varphi}{\partial y} + \nabla\varphi \cdot \nabla d = 0, \quad y = -d,$$

- Kinematic free surface condition

$$\frac{\partial\eta}{\partial t} + \nabla\varphi \cdot \nabla\eta = \frac{\partial\varphi}{\partial y}, \quad y = \eta(\vec{x}, t),$$

- Dynamic free surface condition

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla_{\vec{x},y}\varphi|^2 + g\eta + \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right) = 0, \quad y = \eta(\vec{x}, t).$$

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Lagrangian variational principle

J.C. Luke. *A variational principle for a fluid with a free surface*, J. Fluid Mech., **27**:395–397. 1967

$$\mathcal{L} = \int_{t_1}^{t_2} \int_{\Omega} \rho \mathscr{L} \, d\vec{x} \, dt$$

Classical lagrangian:

$$\mathscr{L} := \int_{-d}^{\eta} \left(\frac{1}{2}|\nabla_{\vec{x},y}\varphi|^2 - gy\right) dy - \sigma\left(\sqrt{1 + |\nabla\eta|^2} - 1\right)$$

Luke's lagrangian (1967):

$$\mathscr{L} := \int_{-d}^{\eta} \left(\frac{1}{2}|\nabla_{\vec{x},y}\varphi|^2 + \varphi_t + gy\right) dy - \sigma\left(\sqrt{1 + |\nabla\eta|^2} - 1\right)$$

Water wave problem - II

Derivation from Luke's lagrangian

$$\delta\mathcal{L} = \delta \int_{t_1}^{t_2} \int_{\Omega} \rho \mathscr{L} \, d\vec{x} \, dt = 0$$

$$\begin{aligned}
\delta\varphi&: \ \Delta\varphi = 0, \quad (\vec{x}, y) \in \Omega \times [-d, \eta], \\
\delta\varphi|_{y=-d}&: \ \frac{\partial\varphi}{\partial y} + \nabla\varphi \cdot \nabla d = 0, \quad y = -d, \\
\delta\varphi|_{y=\eta}&: \ \frac{\partial\eta}{\partial t} + \nabla\varphi \cdot \nabla\eta - \frac{\partial\varphi}{\partial y} = 0, \quad y = \eta(\vec{x}, t), \\
\delta\eta&: \ \frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + g\eta + \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}} \right) = 0, \quad y = \eta(\vec{x}, t).
\end{aligned}$$

- We obtain the water wave problem by varying η and φ

Hamiltonian structure

V.E. Zakharov. *Stability of periodic waves of finite amplitude on the surface of a deep fluid*. J. Appl. Mech. Tech. Phys., **9**, 1968

Canonical variables:

$\eta(\vec{x}, t)$: free surface elevation

$\tilde{\varphi}(\vec{x}, t)$: velocity potential at the free surface
$\tilde{\varphi}(\vec{x}, t) := \varphi(\vec{x}, y = \eta(\vec{x}, t), t)$

Evolution equations:

$$\rho \frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \tilde{\varphi}}, \qquad \rho \frac{\partial \tilde{\varphi}}{\partial t} = -\frac{\delta \mathscr{H}}{\delta \eta},$$

Hamiltonian:

$$\mathcal{H} = \int_{\Omega} \rho \mathscr{H} \, d\vec{x}, \mathscr{H} = \frac{1}{2} g \eta^2 + \sigma \left(\sqrt{1 + |\nabla \eta|^2} - 1 \right) + \int_{-d}^{\eta} \frac{1}{2} |\nabla_{\vec{x},y} \varphi|^2 \, dy$$

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Generalization of the Lagrangian density - I

D. Clamond, D. Dutykh. *Practical use of variational principles for modeling water waves*. In preparation, 2009

$\tilde{\varphi} := \varphi(\vec{x}, y = \eta(\vec{x}, t), t)$: quantity at the free surface

$\check{\varphi} := \varphi(\vec{x}, y = -d(\vec{x}, t), t)$: value at the bottom

Equivalent form of Luke's lagrangian:

$$\mathscr{L} = \tilde{\varphi}\eta_t + \check{\varphi}d_t - \frac{1}{2}g\eta^2 + \frac{1}{2}gd^2 - \int_{-d}^{\eta} \left[\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\varphi_y^2\right] dy$$

Velocity field:

Horizontal velocity: $\vec{u} = \nabla\varphi$

Vertical velocity: $v = \varphi_y$

Generalization of the Lagrangian density - II

D. Clamond, D. Dutykh. *Practical use of variational principles for modeling water waves*. In preparation, 2009

Relaxed variational principle:

$$\mathscr{L} = \tilde{\varphi}\eta_t + \check{\varphi}d_t - \frac{1}{2}g\eta^2 - \int_{-d}^{\eta}\left[\frac{1}{2}(\vec{u}^2 + v^2) + \vec{\mu}\cdot(\nabla\varphi - \vec{u}) + \nu(\varphi_y - v)\right] dy$$

$\vec{\mu}, \nu$: Lagrange multipliers or pseudo-velocity field

After integration by parts:

$$\mathscr{L} = (\eta_t + \tilde{\mu}\cdot\nabla\eta - \tilde{\nu})\tilde{\varphi} + (d_t + \check{\mu}\cdot\nabla d + \check{\nu})\check{\varphi} - \frac{1}{2}g\eta^2 + \int_{-d}^{\eta}\left[\vec{\mu}\cdot\vec{u} - \frac{1}{2}\vec{u}^2 + \nu v - \frac{1}{2}v^2 + (\nabla\cdot\vec{\mu} + \nu_y)\varphi\right] dy$$

Generalization of the Lagrangian density - III

D. Clamond, D. Dutykh. *Practical use of variational principles for modeling water waves*. In preparation, 2009

Variational principle in deep water ($d \to +\infty$)

$$\mathscr{L} = (\eta_t + \tilde{\mu} \cdot \nabla\eta - \tilde{\nu})\tilde{\varphi} - \frac{1}{2}g\eta^2 + \int_{-\infty}^{\eta} \left[\vec{\mu} \cdot \vec{u} - \frac{1}{2}\vec{u}^2 + \nu v - \frac{1}{2}v^2 + (\nabla \cdot \vec{\mu} + \nu_y)\varphi\right] dy$$

Degrees of freedom:

Classical formulation: η, φ

Relaxed formulation: $\eta, \varphi; \vec{u}, v; \vec{\mu}, \nu$

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Shallow water regime

Choice of a simple ansatz in shallow water

Ansatz:

$$\vec{u}(\vec{x},y,t) \approx \bar{u}(\vec{x},t), v(\vec{x},y,t) \approx (y+d)(\eta+d)^{-1}\ \tilde{v}(\vec{x},t)$$
$$\varphi(\vec{x},y,t) \approx \bar{\varphi}(\vec{x},t), \nu(\vec{x},y,t) \approx (y+d)(\eta+d)^{-1}\ \tilde{\nu}(\vec{x},t)$$

Lagrangian density:

$$\mathscr{L} = \bar{\varphi}\eta_t - \frac{1}{2}g\eta^2 + (\eta+d)\left[\bar{\mu}\cdot\bar{u} - \frac{1}{2}\bar{u}^2 + \frac{1}{3}\tilde{\nu}\tilde{v} - \frac{1}{6}\tilde{v}^2 - \bar{\mu}\cdot\nabla\bar{\varphi}\right]$$

Nonlinear Shallow Water Equations:

$$\begin{aligned} h_t + \nabla\cdot[h\bar{u}] &= 0, \\ \bar{u}_t + (\bar{u}\cdot\nabla)\bar{u} + g\nabla h &= 0. \end{aligned}$$

Constraining with free surface impermeability

Constraint:

$$\tilde{\nu} = \eta_t + \bar{\mu} \cdot \nabla \eta$$

- Generalized Serre equations:

$$\begin{aligned} h_t + \nabla \cdot [h\bar{u}] &= 0, \\ \bar{u}_t + \bar{u} \cdot \nabla \bar{u} + g\nabla h + \frac{1}{3} h^{-1} \nabla [h^2 \tilde{\gamma}] &= (\bar{u} \cdot \nabla h) \nabla (h \nabla \cdot \bar{u}) \\ &\quad - [\bar{u} \cdot \nabla (h \nabla \cdot \bar{u})] \nabla h \end{aligned}$$

$$\tilde{\gamma} = \tilde{v}_t + \bar{u} \cdot \nabla \tilde{v} = h\big((\nabla \cdot \bar{u})^2 - \nabla \cdot \bar{u}_t - \bar{u} \cdot \nabla(\nabla \cdot \bar{u})\big)$$

This model cannot be obtained from Luke's lagrangian:

$\delta\bar{\mu}$: $\bar{u} = \nabla\bar{\varphi} - \frac{1}{3}\tilde{v}\nabla\eta \neq \nabla\bar{\varphi}$

Constraining with free surface impermeability

Constraint:

$$\tilde{\nu} = \eta_t + \bar{\mu} \cdot \nabla \eta$$

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$$\tilde{\gamma} = \tilde{v}_t + \bar{u} \cdot \nabla \tilde{v} = h\big((\nabla \cdot \bar{u})^2 - \nabla \cdot \bar{u}_t - \bar{u} \cdot \nabla (\nabla \cdot \bar{u})\big)$$

Solitary wave solution:

$$\eta = a\mathrm{sech}^2 \frac{\kappa}{2}(x - ct), \quad c^2 = g(d + a), \quad (\kappa d)^2 = 3a(d + a)^{-1}$$

Incompressibility and partial potential flow

Ansatz and constraints ($v \neq \varphi_y$):

$$\bar{\mu} = \bar{u}, \tilde{\nu} = \tilde{v}, \bar{u} = \nabla\bar{\varphi}, \tilde{v} = -(\eta + d)\nabla^2\bar{\varphi}$$

$$\mathscr{L} = \bar{\varphi}\eta_t - \frac{1}{2}g\eta^2 - \frac{1}{2}(\eta + d)(\nabla\bar{\varphi})^2 + \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\varphi})^2$$

- Generalized Kaup-Boussinesq equations:

$$\begin{aligned} \eta_t + \nabla \cdot \left[(\eta + d)\nabla\bar{\varphi}\right] + \frac{1}{3}\nabla^2\left[(\eta + d)^3(\nabla^2\bar{\varphi})\right] &= 0, \\ \bar{\varphi}_t + g\eta + \frac{1}{2}(\nabla\bar{\varphi})^2 - \frac{1}{2}(\eta + d)^2(\nabla^2\bar{\varphi})^2 &= 0. \end{aligned}$$

$$\eta = a\cos\kappa(x - ct), \quad c^2 = gd(1 - \frac{1}{3}(\kappa d)^2)$$

Incompressibility and partial potential flow

Ansatz and constraints ($v \neq \varphi_y$):

$$\bar{\mu} = \bar{u}, \tilde{\nu} = \tilde{v}, \bar{u} = \nabla\bar{\varphi}, \tilde{v} = -(\eta + d)\nabla^2\bar{\varphi}$$

$$\mathscr{L} = \bar{\varphi}\eta_t - \frac{1}{2}g\eta^2 - \frac{1}{2}(\eta + d)(\nabla\bar{\varphi})^2 + \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\varphi})^2$$

- Generalized Kaup-Boussinesq equations:

$$\begin{aligned} \eta_t + \nabla\cdot\left[(\eta + d)\nabla\bar{\varphi}\right] + \frac{1}{3}\nabla^2\left[(\eta + d)^3(\nabla^2\bar{\varphi})\right] &= 0, \\ \bar{\varphi}_t + g\eta + \frac{1}{2}(\nabla\bar{\varphi})^2 - \frac{1}{2}(\eta + d)^2(\nabla^2\bar{\varphi})^2 &= 0. \end{aligned}$$

$$\mathscr{H} = \int_\Omega \left\{\frac{1}{2}g\eta^2 + \frac{1}{2}(\eta + d)(\nabla\bar{\varphi})^2 - \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\varphi})^2\right\} d\vec{x}$$

Generalized vertical velocity profile

Ansatz:

$$\varphi \approx \bar{\varphi}(\vec{x},t) \quad \vec{u} \approx \bar{u}(\vec{x},t), \quad v \approx \left(\frac{y+d}{\eta+d}\right)^{\lambda} \tilde{v}(\vec{x},t)$$

$$\vec{\mu} = \vec{u}, \quad \nu = v, \quad \tilde{v} = \eta_t + \tilde{u} \cdot \nabla \eta$$

Lagrangian density ($\beta = (2\lambda + 1)^{-1}$):

$$\mathscr{L} = (\eta_t + [(\eta + d)\bar{u}]_x)\, \tilde{\varphi} - \frac{1}{2} g\eta^2 + \frac{1}{2}(\eta+d)\bar{u}^2 + \frac{1}{2}\beta(\eta+d)\, [\eta_t + \bar{u}\eta_x]^2$$

- Generalized Serre equations (in 1d):

$$\begin{aligned} h_t + [h\bar{u}]_x &= 0, \\ \bar{u}_t + \bar{u}\bar{u}_x + gh_x + \beta h^{-1}[h^2\tilde{\gamma}]_x &= 0, \end{aligned}$$

Generalized Serre equations

Exact solutions: cnoidal wave

$$\eta = a\frac{\mathrm{dn}^2\left(\frac{\kappa}{2}(x-ct)|m\right) - E/K}{1-E/K}, \quad u = \frac{c\eta}{\eta+d}, \quad \frac{\beta^{-1}ga}{(c\kappa d)^2} = 1 - \frac{E}{K}.$$

- dn: elliptic dn-function of Jacobi of parameter m
- $K(m)$: complete elliptic integral of the first kind
- $E(m)$: complete elliptic integral of the second kind

Solitary wave solution:

In the limiting case $m \to 1-$, one gets classical solution:

$$\eta \to a\,\mathrm{sech}^2\left(\frac{\kappa}{2}(x-ct)\right)$$

Optimal choice of the parameter β ?

Deep water approximation

Choice of the ansatz:

$$\{\varphi; \vec{u}; v; \mu; \nu\} \approx \{\tilde{\varphi}; \tilde{u}; \tilde{v}; \tilde{\mu}; \tilde{\nu}\} e^{\kappa(y-\eta)}$$

$$2\kappa\mathscr{L} = 2\kappa\tilde{\varphi}\eta_t - g\kappa\eta^2 + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2 - \tilde{u}\cdot(\nabla\tilde{\varphi} - \kappa\tilde{\varphi}\nabla\eta) - \kappa\tilde{v}\tilde{\varphi}$$

- Deep water Boussinesq equations:

$$\begin{aligned} \eta_t + \frac{1}{2}\kappa^{-1}\nabla^2\tilde{\varphi} - \frac{1}{2}\kappa\tilde{\varphi} &= \frac{1}{2}\tilde{\varphi}[\nabla^2\eta + \kappa(\nabla\eta)^2] \\ \tilde{\varphi}_t + g\eta &= -\frac{1}{2}\nabla\cdot\left[\tilde{\varphi}\nabla\tilde{\varphi} - \kappa\tilde{\varphi}^2\nabla\eta\right] \end{aligned}$$

$$\mathcal{H} = \int_\Omega \left\{\frac{1}{2}g\eta^2 + \frac{1}{4}\kappa^{-1}[\nabla\tilde{\varphi} - \kappa\tilde{\varphi}\nabla\eta]^2 + \frac{1}{4}\kappa\tilde{\varphi}^2\right\} d\vec{x}$$

Deep water approximation

Choice of the ansatz:

$$\{\varphi; \vec{u}; v; \mu; \nu\} \approx \{\tilde{\varphi}; \tilde{u}; \tilde{v}; \tilde{\mu}; \tilde{\nu}\} e^{\kappa(y-\eta)}$$

$$2\kappa\mathscr{L} = 2\kappa\tilde{\varphi}\eta_t - g\kappa\eta^2 + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2 - \tilde{u}\cdot(\nabla\tilde{\varphi} - \kappa\tilde{\varphi}\nabla\eta) - \kappa\tilde{v}\tilde{\varphi}$$

- Deep water Boussinesq equations:

$$\begin{aligned} \eta_t + \frac{1}{2}\kappa^{-1}\nabla^2\tilde{\varphi} - \frac{1}{2}\kappa\tilde{\varphi} &= \frac{1}{2}\tilde{\varphi}[\nabla^2\eta + \kappa(\nabla\eta)^2] \\ \tilde{\varphi}_t + g\eta &= -\frac{1}{2}\nabla\cdot\left[\tilde{\varphi}\nabla\tilde{\varphi} - \kappa\tilde{\varphi}^2\nabla\eta\right] \end{aligned}$$

$$\kappa\eta = \alpha\cos\theta + \frac{1}{2}\alpha^2\left(1 + \frac{25}{12}\alpha^2 + \frac{1675}{192}\alpha^4\right)\cos 2\theta + \cdots + \mathcal{O}(\alpha^8)$$

Finite depth approximation

Bottom impermeability is exactly verified

Choice of the ansatz ($Y := y + d$, $h := \eta + h$):

$$\varphi \approx \frac{\cosh \kappa Y}{\cosh \kappa h}\tilde{\varphi}(\vec{x}, t), \quad \vec{u} \approx \frac{\cosh \kappa Y}{\cosh \kappa h}\tilde{u}(\vec{x}, t), \quad v \approx \frac{\sinh \kappa Y}{\sinh \kappa h}\tilde{v}(\vec{x}, t),$$

$$\vec{\mu} \approx \frac{\cosh \kappa Y}{\cosh \kappa h}\tilde{\mu}(\vec{x}, t), \quad \nu \approx \frac{\sinh \kappa Y}{\sinh \kappa h}\tilde{\nu}(\vec{x}, t)$$

Finite depth lagrangian density:

$$\mathscr{L} = [\eta_t + \tilde{\mu} \cdot \nabla \eta]\tilde{\varphi} - \frac{1}{2}g\eta^2 + [\tilde{\nu}\tilde{v} - \frac{1}{2}\tilde{v}^2]\frac{\sinh(2\kappa h) - 2\kappa h}{2\kappa \cosh(2\kappa h) - 2\kappa}$$

$$+ [\tilde{\mu} \cdot \tilde{u} - \frac{1}{2}\tilde{u}^2 + \tilde{\varphi}\nabla \cdot \tilde{\mu} - \kappa \tanh(\kappa h)\tilde{\varphi}\tilde{\mu} \cdot \nabla \eta]\frac{\sinh(2\kappa h) + 2\kappa h}{2\kappa \cosh(2\kappa h) + 2\kappa}$$

$$+ \frac{1}{2}\tilde{\varphi}\tilde{\nu}\left[\frac{2\kappa h}{\sinh(2\kappa h)} - 1\right]$$

Conclusions and perspectives

Conclusions:

- A novel variational principle has been presented
- Several novel approximate models have been proposed
  - in shallow
  - and deep waters
- Analytical solutions have been obtained when possible

Perspectives:

- Quest for new models in this framework
- Numerical schemes
  - Preserving variational structrure at the discrete level

Thank you for your attention!

http://www.lama.univ-savoie.fr/~dutykh