Simulation numérique des écoulements à surface libre

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Classical water wave problem : physical hypotheses

Trade-off between model complexity and result's quality

1. Flow is incompressible and irrotational : $\frac{D\rho}{Dt} = 0$, $\nabla \times \vec{u} = \vec{0}$,
2. Fluid is ideal : $\nu = 0 \ \frac{m^2}{s}$,
3. Gravity is the only volume force : $\vec{f} = \vec{g}$,
4. Surface tension is neglected[1] : $\sigma = 0 \ \frac{N}{m}$,
5. Pressure is constant on the free surface : $p = p_0$, $z = \eta$,
6. No tangential stresses on the free surface : $\sigma_{xz} = \sigma_{yz} = 0$,
7. Free surface is a graph : $z = \eta(\vec{x}, t)$,
8. Minimal water depth condition : $\eta(\vec{x}, t) + h(\vec{x}, t) \geq d_{min} > 0$

[1] Especially unimportant in long wave limit

1. J. Bona, W. Pritchard & L. Scott, *An Evaluation of a Model Equation for Water Waves.* Phil. Trans. R. Soc. Lond. A, 1981, 302, 457-510

In « Resumé » section :

[...] it was found that the inclusion of a dissipative term was much more important than the inclusion of the nonlinear term, although the inclusion of the nonlinear term was undoubtedly beneficial in describing the observations [...]

2. Boussinesq (1895), Lamb (1932) derived a formula

$$\frac{d\alpha}{dt} = -2\nu k^2 \alpha(t)$$

Mechanisms of dissipation

1. Wave breaking
   - The main effect of wave breaking is the dissipation of energy. This can be modelled by adding dissipative terms in coastal regions where the wave becomes steeper
2. Turbulence
   - For tsunami wave $Re \geq 10^6$, so the flow is turbulent
   - $\Rightarrow$ energy extraction from waves in upper ocean
3. Boundary layers
   - Regions where the viscosity is the most important
     1. free surface boundary layer
     2. bottom boundary layer
4. Molecular viscosity
   - The least important factor for long waves

Energy balance in a fluid flow

- We assume that flow is governed by incompressible Navier-Stokes equations :

$$\begin{aligned} \nabla \cdot \vec{u} &= 0 \\ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} &= \vec{g} - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \tau \end{aligned}$$

- We multiply the second equation by $\vec{u}$ and integrate on control volume Ω :

$$\frac{1}{2} \int\limits_{\Omega} \frac{\partial}{\partial t} \big(\rho |\vec{u}|^2\big)\, d\Omega + \frac{1}{2} \int\limits_{\partial\Omega} \rho |\vec{u}|^2 \vec{u} \cdot \vec{n}\, d\sigma =$$

$$= \int\limits_{\partial\Omega} (-p\mathbb{I} + \tau)\vec{n} \cdot \vec{u}\, d\sigma + \int\limits_{\Omega} \rho \vec{g} \cdot \vec{u}\, d\Omega - \underbrace{\frac{1}{2\mu} \int\limits_{\Omega} \tau : \tau\, d\Omega}_{\mathcal{T}}$$

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Anatomy of dissipation

Estimation of viscous dissipation rate

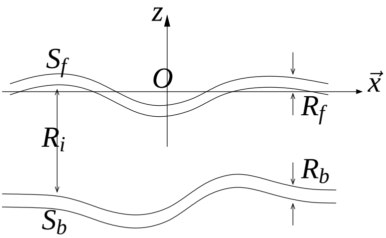


FIG.: Flow regions

- Interior region :
  $\mathcal{T}_{R_i} \sim \frac{1}{\mu}\left(\mu\frac{a}{t_0\ell}\right)^2 \cdot \ell^3 \sim \mu$
- Free surface boundary layer :
  $\mathcal{T}_{R_f} \sim \frac{1}{\mu}\left(\mu\frac{a}{t_0\ell}\right)^2 \cdot \delta\ell^2 \sim \mu^{\frac{3}{2}}$
- Bottom boundary layer :
  $\mathcal{T}_{R_b} \sim \frac{1}{\mu}\left(\mu\frac{a}{t_0\delta}\right)^2 \cdot \delta\ell^2 \sim \mu^{\frac{1}{2}}$

The previous scalings suggest us the following diagram :

$$\underbrace{\mathcal{O}(\mu^{\frac{1}{2}})}_{R_b} \hookrightarrow \underbrace{\mathcal{O}(\mu)}_{R_i \bigcup S_f} \hookrightarrow \underbrace{\mathcal{O}(\mu^{\frac{3}{2}})}_{R_f} \hookrightarrow \underbrace{\mathcal{O}(\mu^2)}_{S_f} \hookrightarrow \ldots$$

1. Consider Navier-Stokes equations
2. Write Helmholtz-Leray decomposition $\vec{u} = \nabla\phi + \nabla \times \vec{\psi}$
3. Express vortical components $\vec{\psi}$ of velocity field in terms of potential ϕ and η using (pseudo) differential (fractional) operators

Kinematic condition :

$$\eta_t = \phi_z + \psi_{2x} - \psi_{1y} \quad \Rightarrow \quad \eta_t = \phi_z + 2\nu\nabla^2\eta$$

Dynamic condition :

$$\phi_t + g\eta + 2\nu\phi_{zz} + \mathcal{O}(\nu^{\frac{3}{2}}) = 0$$

Boundary layer correction

Bottom boundary condition

Ideas of derivation :

1. Consider semi-infinite domain : $z > -h$
2. Use pure Leray decomposition : $\vec{v} = \nabla\phi + \vec{u}, \quad \nabla \cdot \vec{u} = 0$
3. Introduce boundary layer coordinate : $\zeta = \frac{(z+h)}{\delta}$, where $\delta = \sqrt{\nu}$
4. Asymptotic expansion : $\phi = \phi_0 + \delta\phi_1 + \dots$

Bottom condition :

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = -\sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\phi_{zz}|_{z=-h}}{\sqrt{t-\tau}}\, d\tau = -\sqrt{\nu}\mathcal{I}[\phi_{zz}]$$

Nonlocal visco-potential formulation

Resulting governing equations

- Continuity equation

$$\Delta\phi = 0, \quad (x, y, z) \in \Omega,$$

- Kinematic free surface condition

$$\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial \phi}{\partial z} + 2\nu\nabla^2\eta, \quad z = \eta(x, y, t),$$

- Dynamic free surface condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 2\nu\nabla^2\phi, \quad z = \eta(x, y, t).$$

- Kinematic bottom condition

$$\frac{\partial \phi}{\partial z} + \nabla\phi \cdot \nabla h = -\sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\phi_{zz}}{\sqrt{t-\tau}}\, d\tau, \quad z = -h(x, y),$$

Linear dispersion relation

For full equations. In Boussinesq equations replace $\tanh(|\vec{k}|h)$ by corresponding rational approximation

We look for the following periodic solutions :

$\phi(\vec{x}, z, t) = \varphi(z)\ e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, $\eta(\vec{x}, t) = \eta_0\ e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ with $h \equiv const$

- If we do not modify bottom boundary condition :

$$\omega(|\vec{k}|) = \sqrt{g|\vec{k}|\tanh(|\vec{k}|h)} - 2i\nu|\vec{k}|^2$$

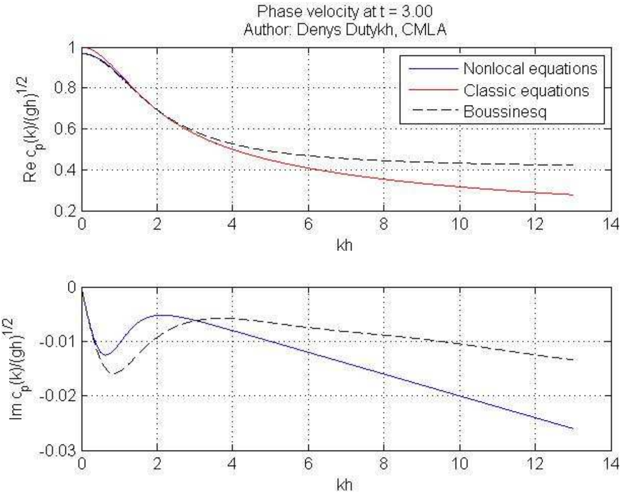
- Dispersion relation with nonlocal term :

$$(i\omega - 2\nu|\vec{k}|^2)^2 + g|\vec{k}|\tanh(|\vec{k}|h) = |\vec{k}|\sqrt{\frac{i\nu}{\omega}}\Big((i\omega - 2\nu|\vec{k}|^2)\tanh(|\vec{k}|h) + g|\vec{k}|\Big)\,\mathrm{erf}(\sqrt{-i\omega t})$$

- Time dependent dispersion : $\omega = \omega(|\vec{k}|, t)$

Dependence of phase velocity on time

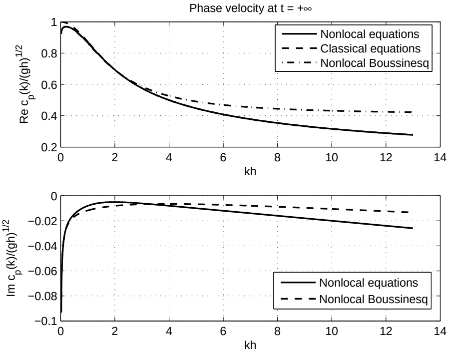
With nonlocal term : $\omega = \omega(|\vec{k}|, t)$



Phase velocity at infinite time

We take analytic limit as $t \to +\infty$

$$D(\omega_\infty, k) := (i\omega_\infty - 2\nu k^2)^2 + gk\tanh(kh) - \sqrt{\frac{\nu}{\omega_\infty}} k e^{i\frac{\pi}{4}} \left((i\omega_\infty - 2\nu k^2)^2 \tanh(kh) + gk\right) \equiv 0$$



Long wave approximation

1. Nonlocal Boussinesq equations :
   - Mass conservation :

$$\eta_t+\nabla\cdot((h+\eta)\vec{u})+A_\theta h^3\nabla^2(\nabla\cdot\vec{u}) = 2\nu\Delta\eta+\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{\nabla\cdot\vec{u}}{\sqrt{t-\tau}}\,d\tau$$

   - Horizontal momentum :

$$\vec{u}_t+\frac{1}{2}\nabla|\vec{u}|^2+g\nabla\eta-B_\theta h^2\nabla(\nabla\cdot\vec{u}_t) = 2\nu\Delta\vec{u}$$

2. Nonlocal KdV equation :

$$\eta_t+\sqrt{\frac{g}{h}}\Big((h+\frac{3}{2}\eta)\eta_x+\frac{1}{6}h^3\eta_{xxx}-\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{\eta_x}{\sqrt{t-\tau}}\,d\tau\Big) = 2\nu\eta_{xx}$$

Solitary wave attenuation

Effect of nonlocal term on the amplitude

Numerical solution to nonlocal Boussinesq equations

- Solitary wave initial condition
- Fourier-type spectral method
- Comparison between :
  1. Classical Boussinesq equations
  2. Local dissipative terms
  3. Local + nonlocal dissipation

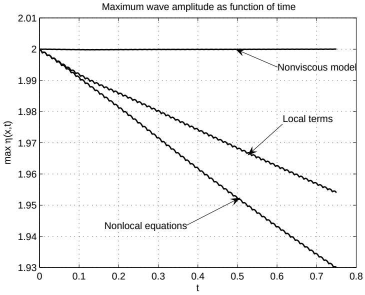


FIG.: Soliton amplitude

Numerical solution to nonlocal Boussinesq equations

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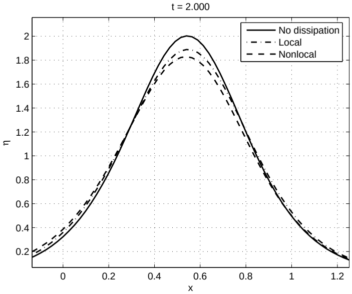


FIG.: Zoom on soliton crest

- Consider nonlocal dissipative Airy equation

$$\eta_t + \sqrt{\frac{g}{h}}\Big(h\eta_x + \frac{1}{6}h^3\eta_{xxx} - \sqrt{\frac{\nu}{\pi}}\int\limits_0^t \frac{\eta_x}{\sqrt{t-\tau}}\,d\tau\Big) = 2\nu\eta_{xx}$$

- Special form of solutions

$$\eta(x,t) = \mathcal{A}(t)e^{ik\xi}, \quad \xi = x - \sqrt{gh}t, \quad \mathcal{A}(t) \in \mathbb{C}$$

Integro-differential equation :

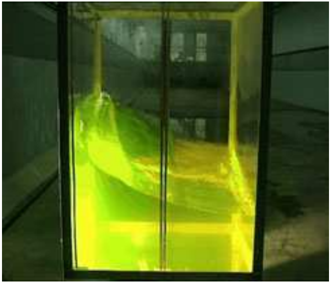
$$\frac{d|\mathcal{A}|^2}{dt} + 4\nu k^2|\mathcal{A}(t)|^2 + ik\sqrt{\frac{g\nu}{\pi h}}\int\limits_0^t \frac{\bar{\mathcal{A}}(t)\mathcal{A}(\tau) - \mathcal{A}(t)\bar{\mathcal{A}}(\tau)}{\sqrt{t-\tau}}\,d\tau = 0$$

Contents

Physical phenomena

Two applications which motivated this study

- Wave sloshing in Liquefied Natural Gas (LNG) carriers



- Wave impacts on coastal structures

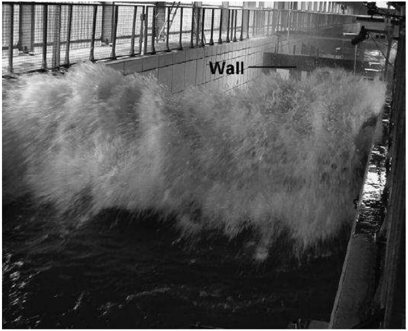
FIG.: GWK, Hannover

Wave impacts on a wall

Ref : Bullock, Obhrai, Peregrine, Bredmose, 2007

Impacts classification :

- low-aeration : the water adjacent to the wall contains typically 5% of air
- high-aeration : higher level of entrained air with clear evidence of entrapment



Main results of the experimental study

Ref : Bullock, Obhrai, Peregrine, Bredmose, 2007

- Low-aeration impact
  - temporary and spatially localised pressure impulse
- High-aeration impact
  - less localised pressure spike with a longer rise time, fall time and duration
  - peak values of the pressure are lower

Conclusion :

« Even when the pressures during a high-aeration impact are lower, the fact that the impact is less spatially localised and lasts longer may well lead to a higher total impulse »

Influence of aeration

Ideas for mathematical modelling

For low-aeration water wave impact ($\alpha_g \approx 0.05$) :

- Sound speed drops down to $\approx 54\frac{m}{s}$
- Compressible effects are very important
  - Mach number is not tiny anymore
- CFL condition is not so severe
  - Explicit in time scheme

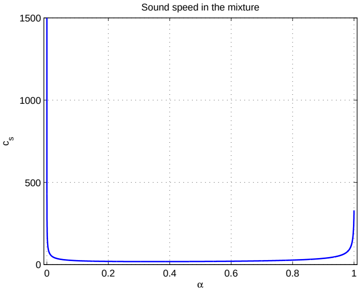


FIG.: Sound speed in the air/water mixture

Mass conservation for each phase :

$$\partial_t(\alpha^{\pm}\rho^{\pm}) + \nabla \cdot (\alpha^{\pm}\rho^{\pm}\vec{u}) = 0,$$

Momentum equation :

$$\partial_t(\rho\vec{u}) + \nabla \cdot \left(\rho\vec{u} \otimes \vec{u} + p\mathbb{I}\right) = \rho\vec{g},$$

Energy conservation :

$$\partial_t\left(\rho E\right) + \nabla \cdot \left(\rho H\vec{u}\right) = \rho\vec{g} \cdot \vec{u},$$

$$\alpha^+ + \alpha^- = 1,\ \rho := \alpha^+\rho^+ + \alpha^-\rho^-,\ H := E + \frac{p}{\rho},\ E := e + \frac{1}{2}|\vec{u}|^2.$$

Two-phase homogenous model - II

Equation of state

- Ideal gas law for light fluid :

$$\begin{aligned} p^- &= (\gamma - 1)\rho^- e^-, \\ e^- &= c_v^- T^-, \end{aligned}$$

- Tait's law for heavy fluid :

$$\begin{aligned} p^+ + \pi_0 &= (\mathcal{N} - 1)\rho^+ e^+, \\ e^+ &= c_v^+ T^+ + \frac{\pi_0}{\mathcal{N}\rho^+}, \end{aligned}$$

where γ, $c_v^\pm$, π_0, $\mathcal{N}$ are constants
Additional assumption : Two phases are in thermodynamic equilibrium :

$$p := p^+ = p^-, \qquad T := T^+ = T^-$$

Motivation for the choice of this model

Trade-off between model complexity and accuracy of the results

Main reasons

- This model is **hyperbolic**
- There is **no special treatment** of the free surface
- We have only **four** equations in 1D
- Equations do not contain **nonconservative products**
- Eigenvalues and eigenvectors can be computed **analytically**

$\Rightarrow$ computation is not expensive

We believe that this model gives qualitatively correct results for the flow and right impact pressure

Limit to the classical free surface model

For illustration we take the barotropic case

- Consider the following system :

$$\begin{aligned} \partial_t(\alpha^\pm \rho^\pm) + \nabla \cdot (\alpha^\pm \rho^\pm \vec{u}) &= 0, \\ \partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p &= \rho \vec{g}, \end{aligned}$$

- Equation of state : $p = p(\rho^+) = p(\rho^-)$
- Speeds of sound are given by $(c^\pm)^2 := \frac{\partial p}{\partial \rho^\pm}$
- We define the following quantities :

$$\gamma^\pm := \frac{\rho^\pm (c^\pm)^2}{p}, \qquad \bar{\gamma} := \alpha^+ \gamma^- + \alpha^- \gamma^+, \qquad \delta^\pm \gamma = \gamma^\pm - \gamma^\mp$$

Limit to a discontinous two-phase system

Two phases separated by an interface

- One can derive the following equation for $\alpha^{\pm}$:

$$\alpha_t^{\pm} + (\vec{u} \cdot \nabla)\alpha^{\pm} + \alpha^{+}\alpha^{-}\frac{\delta^{\pm}\gamma}{\bar{\gamma}}\nabla \cdot \vec{u} = 0 \qquad (1)$$

- We separate the phases : $\alpha^{-} = H(z - \eta(\vec{x}, t))$
  - It follows that : $\alpha^{+}\alpha^{-} = 0$

- Substituting in (1) gives

$$\eta_t + (u, v) \cdot \nabla_x \eta = w$$

- Governing equations become

$$\begin{aligned} \rho_t + (\vec{u} \cdot \nabla)\rho + \rho\nabla \cdot \vec{u} &= 0, \\ \vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} + \frac{\nabla p}{\rho} &= \vec{g}. \end{aligned}$$

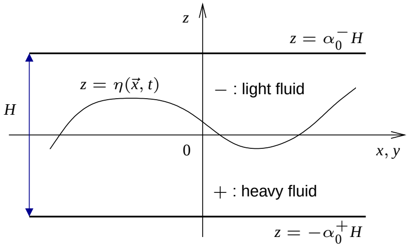


FIG.: Sketch of the flow.

System of balance laws

Rewrite governing equations :

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathcal{F}(w) = \mathcal{S}(\vec{x}, t, w),$$

Integrate them over control volume :

$$\frac{d}{dt} \int_K w \, d\Omega + \int_{\partial K} \mathcal{F}(w) \cdot \vec{n}_{KL} \, d\sigma = \int_K S(w) \, d\Omega$$

Introduce cell averages :

$$w_K(t) := \frac{1}{\mathrm{vol}(K)} \int_K w(\vec{x}, t) \, d\Omega$$

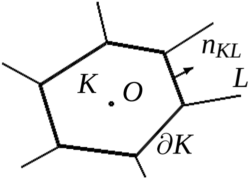


FIG.: Control volume

How to express $(\mathcal{F} \cdot \vec{n})|_{\partial K}$ in terms of $\{w_K\}_{K \in \Omega}$?

Finite volumes scheme

Volumes Finis à Flux Caractéristique (VFFC)

Use numerical flux of VFFC scheme to discretize advection operator :

$$\Phi(w_K, w_L; \vec{n}_{KL}) = \frac{\mathcal{F}_n(w_K) + \mathcal{F}_n(w_L)}{2} - U(\mu; \vec{n}_{KL}) \frac{\mathcal{F}_n(w_L) - \mathcal{F}_n(w_K)}{2}$$

where μ is a mean state

$$\mu := \frac{\mathrm{vol}(K) w_K + \mathrm{vol}(L) w_L}{\mathrm{vol}(K) + \mathrm{vol}(L)}$$

and $U(\mu; \vec{n}_{KL})$ is the sign matrix

$$U := \mathrm{sign}(\mathbb{A}_n) \equiv \mathbf{R}\, \mathrm{sign}(\Lambda) \mathbf{R}^{-1}, \qquad \mathbb{A}_n := \frac{\partial(\mathcal{F} \cdot \vec{n})(w)}{\partial w}$$

Remark : Since, the advection operator is relatively simple, U can be computed analytically.

Second-order sequel : MUSCL scheme

Monotone Upstream-centered Schemes for Conservation Laws

- We find our solution in class of affine by cell functions :

$$w_K(\vec{x}, t) := \bar{w}_K + (\nabla w)_K(\vec{x} - \vec{x}_0)$$

  - x_0 is the barycenter of the cell K

- Gradient reconstruction :
  - Least-squares method
  - Green-Gauss reconstruction
- Slope limiter :
  - Barth-Jespersen (1989)
  - no limiter at all

Time integration : SSP-RK3(4) scheme

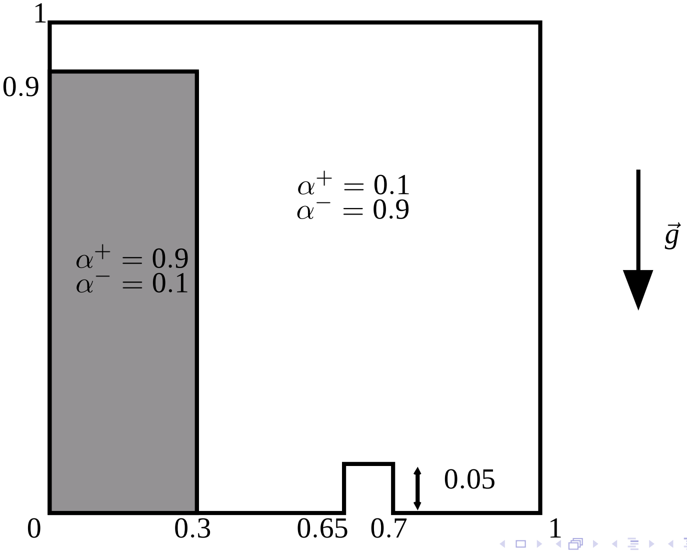
Ref : Spiteri & Ruuth (2002), SIAM J. Numer. Anal.

Third order 4-stage scheme with $CFL = 2$:

$$\begin{aligned}
u^{(1)} &= u^{(n)} + \frac{1}{2}\Delta t L(u^{(n)}), \\
u^{(2)} &= u^{(1)} + \frac{1}{2}\Delta t L(u^{(1)}), \\
u^{(3)} &= \frac{2}{3}u^{(n)} + \frac{1}{3}u^{(2)} + \frac{1}{6}\Delta t L(u^{(n)}), \\
u^{(n+1)} &= u^{(3)} + \frac{1}{2}\Delta t L(u^{(3)}),
\end{aligned}$$

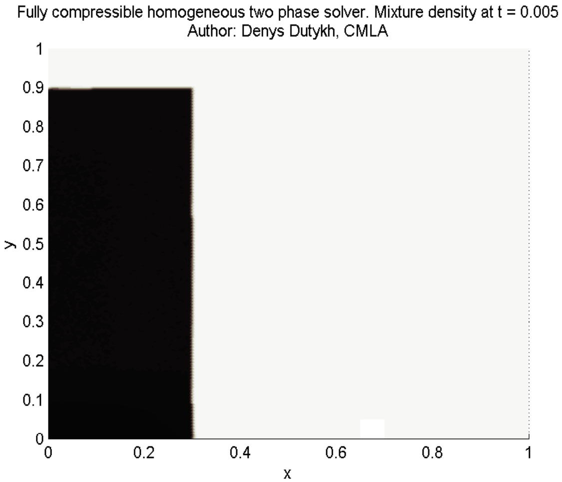
Water column test case - I

Geometry and description of the test case



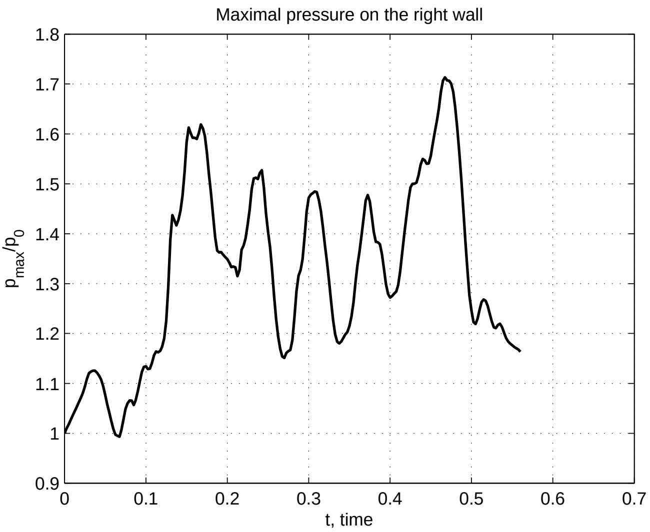
Water column test case - II

Gravity acceleration $g = 100 m/s^2$,in heavy fluid $\alpha^+ = 0.9$, in light fluid $\alpha^+ = 0.1$



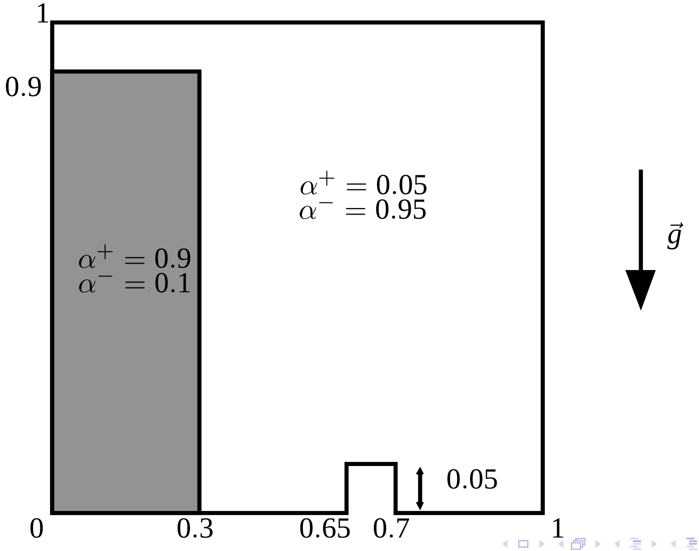
Maximal pressure on the right wall

as a function of time $t \longmapsto \max_{(x,y)\in 1\times[0,1]} p(x,y,t)$



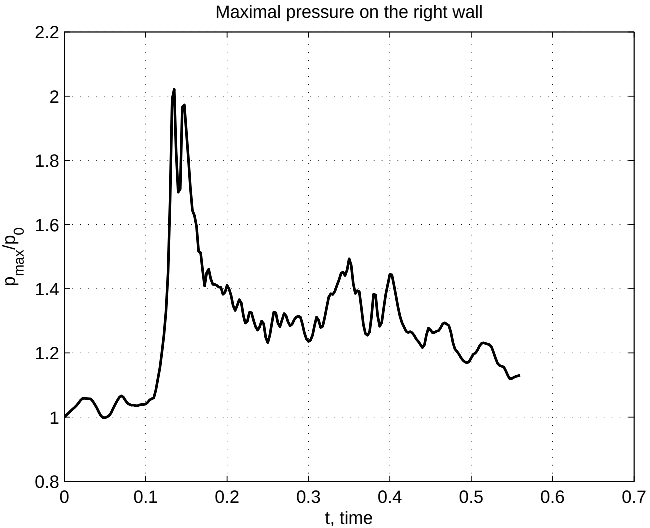
Water column test case - III

Lighter gas case



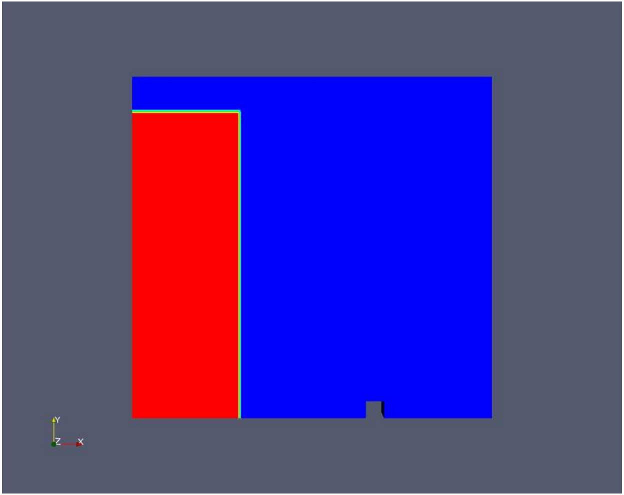
Maximal pressure on the right wall

as a function of time $t \longmapsto \max_{(x,y)\in 1\times[0,1]} p(x,y,t)$



Incompressible computation

Two-fluids Navier-Stokes equations solver : OpenFOAM



Impact pressures at the right wall

Comparison between compressible and incompressible models

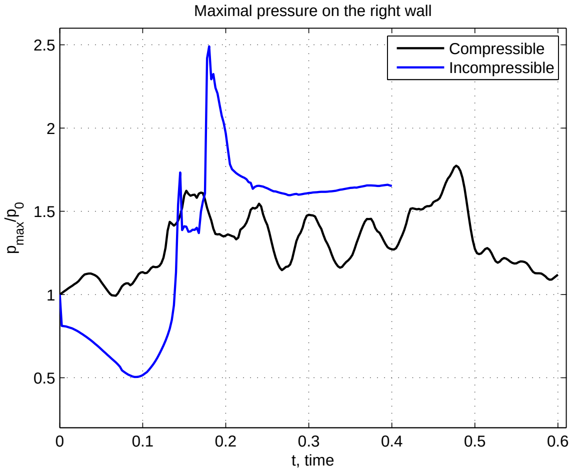


FIG.: Gas volume fraction $\alpha_g = 0.9$

Impact pressures at the right wall

Comparison between compressible and incompressible models

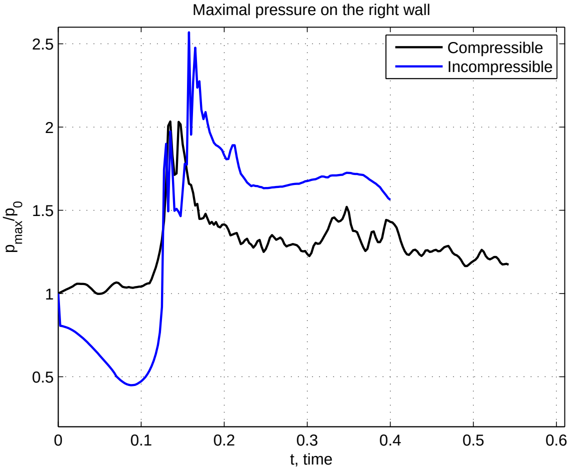


FIG.: Gas volume fraction $\alpha_g = 0.95$

Perspectives

Directions for future work

Physics :

- Quantitative comparison with 6-equations model
- Parametric study : aeration influence on impact pressures
- Wave breaking influence

Mathematics :

- Formal justification of 4-equations model

Numerics :

- Towards pure phases computations
  - Implicit time stepping
  - Low Mach number problem

Other research directions

If I had more time. . .

- Tsunami waves modelling
  - Tsunami generation by earthquakes
  - Seismology/hydrodynamics coupling
  - Sediment layer effect
  - Runup simulation
- Boussinesq equations
  - Derivation of long wave models
  - Numerical solution with FV : development of the code *VOLNA*

Thank you for your attention !

http://www.cmla.ens-cachan.fr/~dutykh