On shallow capillary-gravity waves

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"*Modelling, theory and numerical approximation of nonlinear waves*"
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- Didier CLAMOND: Professor (LJAD)
  Université de Nice Sophia Antipolis
- André GALLIGO: Emiritus Professor
  Université de Nice Sophia Antipolis

- Consider the dispersion relation:

$$\omega^2 = k\left(g + \frac{\sigma}{\rho}k^2\right)$$

ω, k: wave frequency, wave number
g, σ: gravity acceleration, surface tension

- **Gravity wave** regime ($k \ll 1$):

$$\omega^2 = gk$$

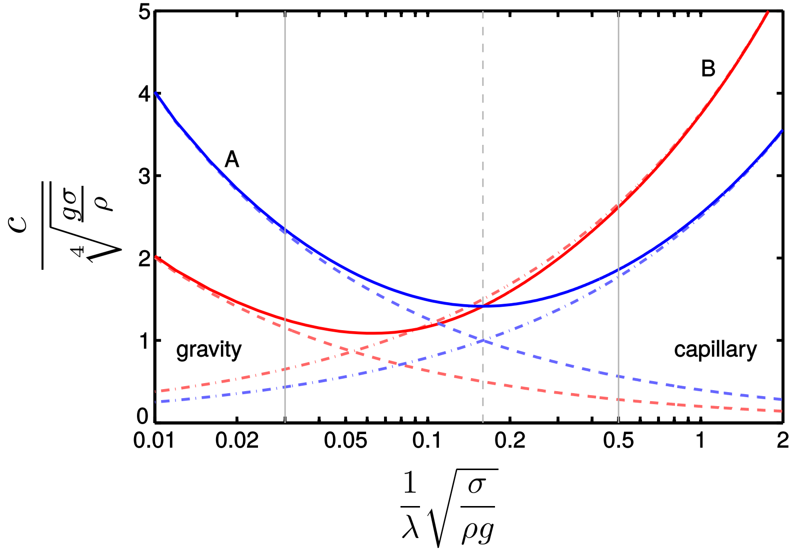
- **Capillary wave** regime ($k \gg 1$):

$$\omega^2 = \frac{\sigma}{\rho}k^3$$

- **Capillary-gravity** regime ($k \propto 1$)

Capillary-gravity waves

Deep water case (for the sake of simplicity)



- Phase velocity minimum:

$$\lambda_c = 2\pi\sqrt{\frac{\sigma}{\rho g}}$$

- For the air-water interface:

$$\lambda_c \approx 1.7 \text{ cm}$$

- In vicinity of λ_c both effects have to be taken into account!

Serre–(Green–Naghdi) equations

Shallow water equations

Main constitutive assumptions:

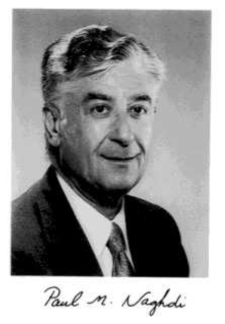
- Free surface is a graph: $y = \eta(x, t)$
- Long wave (or *equivalently* the water is shallow):

$$\lambda \gg d$$

- Nonlinearity is finite: $\varepsilon \equiv a/d \propto \mathcal{O}(1)$

Credits:

- Lord RAYLEIGH (1876) [1] (only *steady* version)
- F. SERRE (1953) [2]
- C. SU & C. GARDNER (1969) [3]
- A. GREEN & P. NAGHDI (1976) [4]
- E. PELINOVSKY & ZHELEZNYAK (1985) [5]



- Governing equations (mass conservation ($\star$), momentum ($*$)):

$$h_t + [h\bar{u}]_x = 0, \qquad (\star)$$

$$\bar{u}_t + \bar{u}\,\bar{u}_x + g\,h_x + \tfrac{1}{3}h^{-1}\partial_x\Big[h^2\,\tilde{\gamma}\Big] = \tau\Big[h_x\Big(1+h_x^2\Big)^{-1/2}\Big]_{xx} \qquad (*)$$

- Vertical acceleration:

$$\tilde{\gamma} \equiv h(\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}) = 2\,h\,\bar{u}_x^2 - h[\bar{u}_t + \bar{u}\,\bar{u}_x]_x$$

- quasi-Conservative form:

$$[h\,\bar{u}]_t + \Big[h\,\bar{u}^2 + \tfrac{1}{2}g\,h^2 + \tfrac{1}{3}h^2\,\tilde{\gamma} - \tau\,R\Big]_x = 0$$

- Surface tension ($\tau \equiv \frac{\sigma}{\rho}$):

$$R = h\,h_{xx}\Big(1+h_x^2\Big)^{-3/2} + \Big(1+h_x^2\Big)^{-1/2},$$

Serre's equations: the conservation laws

- Momentum conservation:

$$\Big[h\bar{u} - \tfrac{1}{3}(h^3\bar{u}_x)_x\Big]_t + \Big[h\bar{u}^2 + \tfrac{1}{2}gh^2 - \tfrac{1}{3}2h^3\bar{u}_x^2 - \tfrac{1}{3}h^3\bar{u}\bar{u}_{xx} - h^2h_x\bar{u}\bar{u}_x - R\Big]_x = 0$$

- Tangential velocity at the free surface:

$$\Big[\bar{u} - \frac{(h^3\bar{u}_x)_x}{3\,h}\Big]_t + \Big[\tfrac{1}{2}\bar{u}^2 + gh - \tfrac{1}{2}h^2\,\bar{u}_x^2 - \frac{\bar{u}\,(h^3\bar{u}_x)_x}{3\,h} - \frac{\tau\,h_{xx}}{(1+h_x^2)^{3/2}}\Big]_x = 0$$

- Energy conservation:

$$\Big[\tfrac{1}{2}h\,\bar{u}^2 + \tfrac{1}{6}h^3\,\bar{u}_x^2 + \tfrac{1}{2}g\,h^2 + \tau\sqrt{1+h_x^2}\Big]_t +$$

$$\Big[\Big(\tfrac{1}{2}\bar{u}^2 + \tfrac{1}{6}h^2\,\bar{u}_x^2 + gh + \tfrac{1}{3}h\gamma - \frac{\tau R}{h}\Big)h\,\bar{u} + \tau\,\bar{u}\sqrt{1+h_x^2} + \frac{\tau\,h\,h_x\,\bar{u}_x}{\sqrt{1+h_x^2}}\Big]_x = 0$$

Serre–CG equations: travelling waves

$\mathrm{Fr} = c^2/gd$, $\mathrm{Bo} = \tau/gd^2$, $\mathrm{We} = \mathrm{Bo}\,/\,\mathrm{Fr} = \tau/c^2 d$

- Mass conservation: $\bar{u} = -cd \;/\; h$, $d =< h >= \frac{1}{2\ell}\int_{-\ell}^{\ell} h\,\mathrm{d}x$
- Momentum conservations lead:

$$\frac{\mathrm{Fr}\,d}{h} + \frac{h^2}{2\,d^2} + \frac{\tilde{\gamma}h^2}{3gd^2} - \frac{\mathrm{Bo}\,hh_{xx}}{\left(1+h_x^2\right)^{\frac{3}{2}}} - \frac{\mathrm{Bo}}{\left(1+h_x^2\right)^{\frac{1}{2}}} = \mathrm{Fr} + \frac{1}{2} - \mathrm{Bo} + K_1$$

- Tangential velocity:

$$\frac{\mathrm{Fr}\ d^2}{2\,h^2} + \frac{h}{d} + \frac{\mathrm{Fr}\ d^2\,h_{xx}}{3\,h} - \frac{\mathrm{Fr}\ d^2\,h_x^2}{6\,h^2} - \frac{\mathrm{Bo}\ d\,h_{xx}}{\left(1+h_x^2\right)^{\frac{3}{2}}} = \frac{\mathrm{Fr}}{2} + 1 + \frac{\mathrm{Fr}\ K_2}{2}$$

$$\tilde{\gamma}/g \;=\; \mathrm{Fr}\,d^3\,h_{xx}/h^2 \;-\; \mathrm{Fr}\,d^3\,h_x^2/h^3$$

- Integration constants:

$$K_2 \;=\; \left\langle\,\frac{\left(3+h_x^2\right)d^2}{3\,h^2} - 1\,\right\rangle$$

$$K_1 + \tfrac{1}{2} + \mathrm{Fr} - \mathrm{Bo} \;=\; \left\langle\,\frac{\mathrm{Fr}\ d^2}{h^2} + \frac{1}{2} - \frac{\mathrm{Bo}\ d\,/\,h}{\left(1+h_x^2\right)^{\frac{1}{2}}}\,\right\rangle\Big/\left\langle\,\frac{d}{h}\,\right\rangle.$$

- Combination of two equations:

$$\frac{\mathrm{Fr}\,d}{2\,h} - \frac{h^2}{2\,d^2} - \frac{\mathrm{Fr}\,d\,h_x^2}{6\,h} - \frac{\mathrm{Bo}}{(1+h_x^2)^{\frac{1}{2}}} + \frac{(\mathrm{Fr}+2+\mathrm{Fr}\,K_2)\,h}{2\,d} = \mathrm{Const}$$

- Consider solitary waves ($K_1 = K_2 \equiv 0$):

$$F(h,h') \equiv \frac{\mathrm{Fr}\,h'^2}{3} + \frac{2\,\mathrm{Bo}\;h/d}{\left(1+h'^2\right)^{\frac{1}{2}}} - \mathrm{Fr} + \frac{(2\mathrm{Fr}+1-2\mathrm{Bo})\,h}{d} - \frac{(\mathrm{Fr}+2)\,h^2}{d^2} + \frac{h^3}{d^3} = 0$$

- Property (a discrete symmetry): $h(x) \equiv h(-x)$
- At the crest of a regular wave: $h(0) = d + a$, $h'(0) = 0$

$$\mathrm{Fr} = 1 + a/d$$

- Consider an implicit ODE (surface $\mathcal{M} \subseteq \mathbb{R}^3$):

$$F(x, y, p) = 0, \qquad p := \frac{\mathrm{d}y}{\mathrm{d}x}.$$

- Coordinates (x, y, p) live in the 1-jet space:
  - Two smooth functions $y(x)$ and $z(x)$ belong to the same k-jet in x_0 if and only if

$$y(x) - z(x) = o\bigl(|x - x_0|^k\bigr)$$

- Consider the projection map:

$$\pi : \mathcal{M} \to \mathbb{R}^2 \quad (x, y, p) \mapsto (x, y)$$

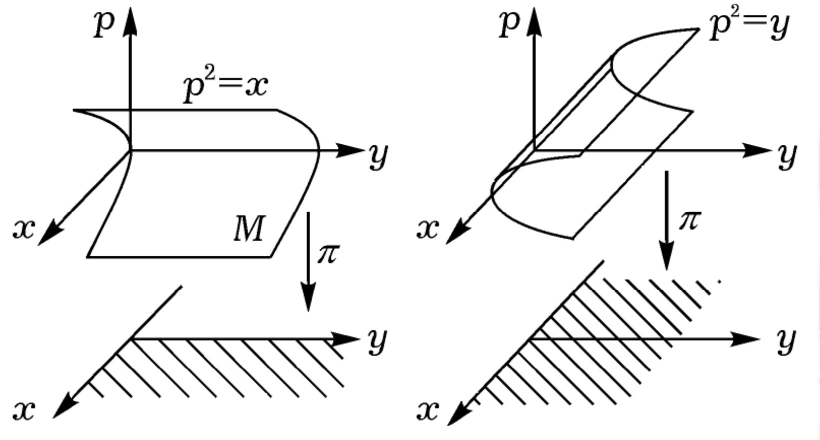
Definition:

A point $(x, y, p) \in \mathcal{M}$ is regular if π is a diffeomorphism in this point.

Reference [6]:

Arnold, V. I. (1996). *Geometrical Methods in the Theory of Ordinary Differential Equations* (pp. 351). New York: Springer–Verlag.

- Criminant and discriminant curves:



Normal form of an implicit ODE

Regular points ($F'_p(x_0, y_0, p_0) \neq 0$):

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Theorem ([6])

Let (x_0, y_0, p_0) be a regular critical point[a] of the equation $F(x, y, p) = 0$. Then, there exists a diffeomorphism of a neighbourhood of the point $(x_0, y_0) \in \mathbb{R}^2$ on a neighbourhood of the point $(X, Y) = (0, 0)$ transforming locally the equation $F(x, y, p) = 0$ to $P^2 = X$, where $P = \frac{\mathrm{d}Y}{\mathrm{d}X}$.

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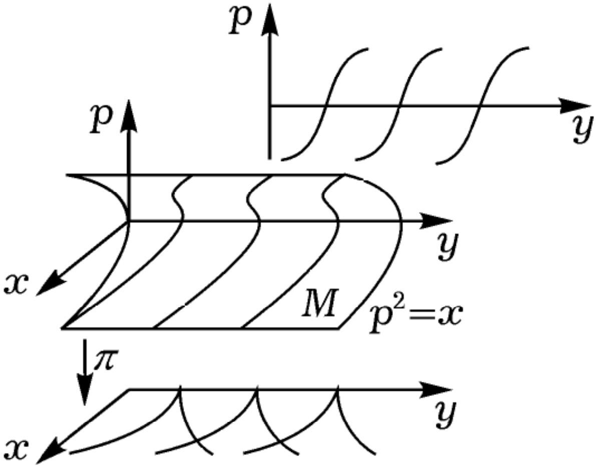
Remarks:

- René THOM found "only" the normal form $P^2 = X\,E(X, Y)$
- Yu. A. BRODSKY improved his proof to obtain $P^2 = X$
- $\Longrightarrow$ Integral curves are locally diffeomorphic to $Y = \frac{2}{3}X^{3/2} + C$

- Normal form analysis suggests:



- Two-parameter $(\mathrm{Fr}, \mathrm{Bo})$ family of real algebraic curves in $\mathbb{R}^2$:

$$F_{\mathrm{Fr}, \mathrm{Bo}}(h, p) := \frac{\mathrm{Fr}}{3} p^2 + 2 \frac{\mathrm{Bo}\, h}{(1 + p^2)^{\frac{1}{2}}} + \\ - \mathrm{Bo} + (2\mathrm{Fr} - 2\mathrm{Bo} + 1) h - (\mathrm{Fr} + 2) h^2 + h^3 = 0$$

- With asymptotic behaviour at $x \to \infty$ $(p := h')$:

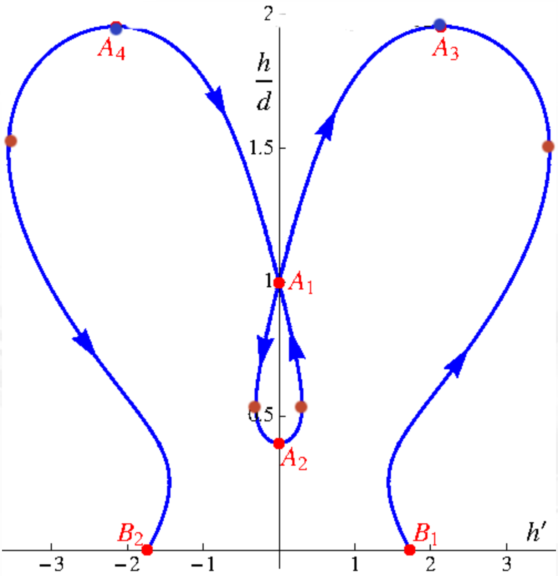
$$h(\infty) = 1, \quad p(\infty) = 0$$

Typical workflow with a parametrized curve:

- Find multiple points (with horizontal tangent): $\partial_p F_{\mathrm{Fr}, \mathrm{Bo}} = 0$
- Find points with vertical tangent: $\partial_h F_{\mathrm{Fr}, \mathrm{Bo}} = 0$
- Decompose it into oriented branches ($p \gtrless 0$, $h \nearrow \searrow 0$)
- Study the family of algebraic curves $F_{\mathrm{Fr}, \mathrm{Bo}}(h, p) \in \mathbb{R}^2$ with certified topology methods [7]

Phase-space analysis: depression wave

A particular example for $\mathrm{Fr} = 0.4$, $\mathrm{Bo} = 0.9 > 1/3$



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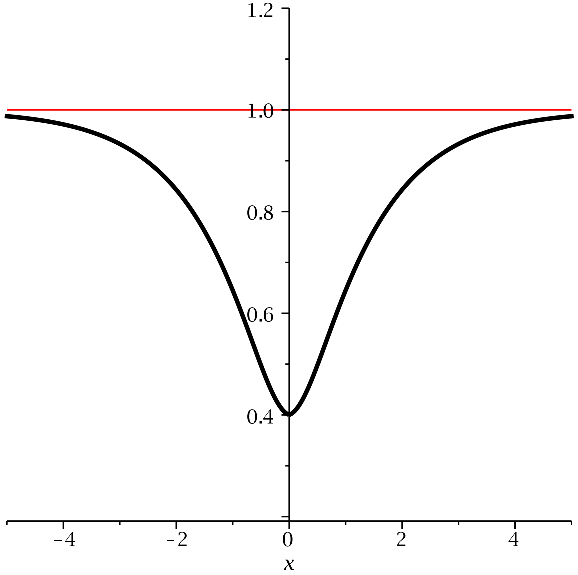

Phase-space analysis: peakon of depression

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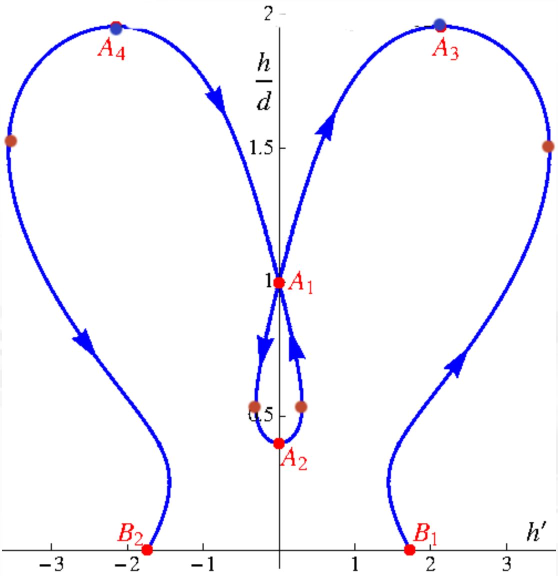

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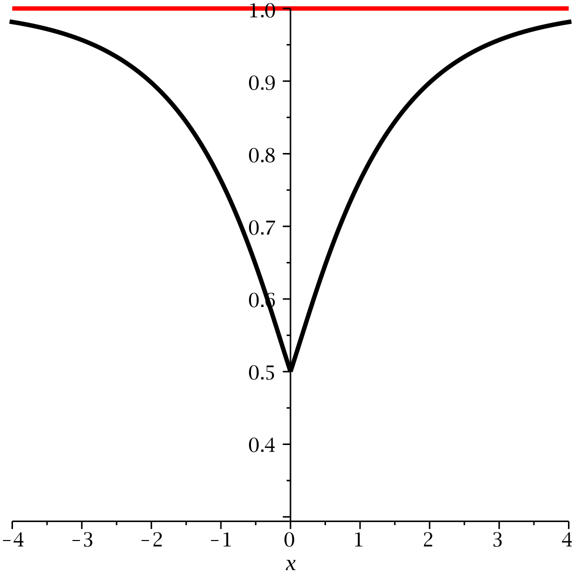

Phase-space analysis: multi-peakon of depression

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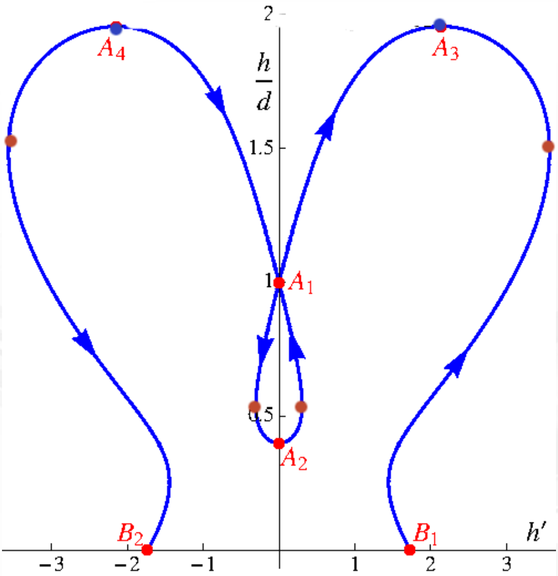

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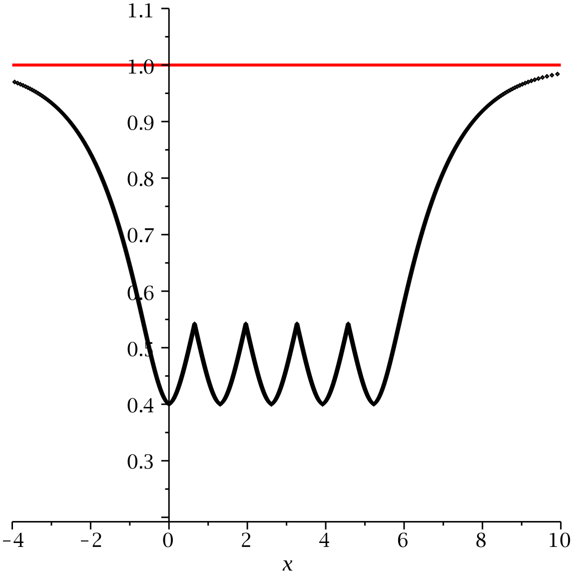


- Consider two polynomials $P, Q \in \mathbb{R}[x]$:

$$\deg P = n, \quad \deg Q = m$$

$$P(x) \;=\; a_0 x^n \;+\; \ldots, \qquad Q(x) \;=\; b_0 x^m \;+\; \ldots$$

- Resultant of two polynomials is

$$\boldsymbol{R}(P, Q) \;:=\; a_0^m b_0^n \prod_{i=1}^{n} \prod_{j=1}^{m} (r_i \;-\; s_j)$$

where r_i are zeros of $P(x)$ and s_j are zeros of $Q(x)$.

- Resultant is the determinant of the Sylvester matrix!

- Discriminant of a polynomial equation $P(x) = 0$ is

$$\mathcal{D}(P) \;:=\; (-1)^{\frac{n(n-1)}{2}} \boldsymbol{R}(P, P')$$

- Quadratic polynomials:

$$\mathcal{D}(a_2x^2 + a_1x + a_0) = a_1^2 - 4a_2a_0$$

- Cubic polynomials:

$$\mathcal{D}(a_3x^3 + a_2x^2 + a_1x + a_0) = a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3$$

Discriminant depends on the degree bound!

Reference [8]:

Gelfand, I. M., Kapranov, M. M., & Zelevinsky, A. V. (1994). *Discriminants, resultants and multidimensional determinants* (pp. 516). Boston: Birkhäuser.

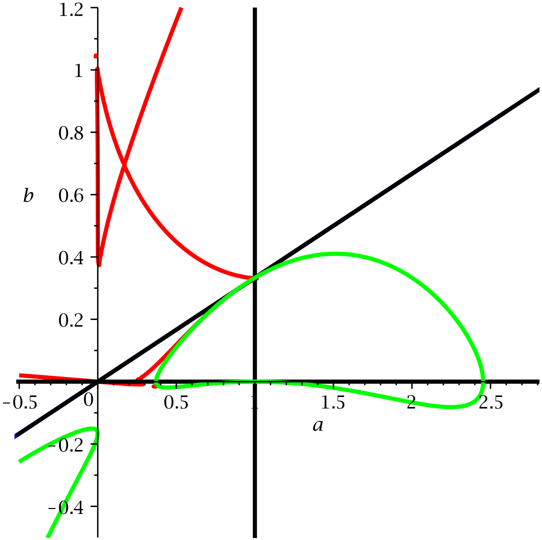
- Points with horizontal tangent satisfy:
  - $F_{\mathrm{Fr},\,\mathrm{Bo}}(p, h) = 0$
  - $\partial_p F_{\mathrm{Fr},\,\mathrm{Bo}}(p, h) = 0$
- The 2nd equation can be solved analytically:
  - $p = 0$
  - $\mathrm{Fr}\,(p^2 + 1)^3 = 9\mathrm{Bo}^2 h^2$
- To avoid square roots, change of variables:
  - $p^2 = y^2 - 1$, $y \geq 1$
  - Wave height can be expressed as $h = \frac{\mathrm{Fr}}{3\mathrm{Bo}} y^3$
- Polynomial equation in y:

$$f := \mathrm{Fr}^2 y^9 - (3\mathrm{Fr} - 2)\mathrm{Fr}\,\mathrm{Bo}\,y^6 + 9\mathrm{Bo}^2(1 + 2\mathrm{Fr} - 2\mathrm{Bo})y^3 + 27\mathrm{Bo}^3 y^2 - 36\mathrm{Bo}^3$$

- Adapted tool to describe the real roots in $(\mathrm{Fr}, \mathrm{Bo})$ space: discriminant locus! $\mathcal{D} = (\mathrm{Fr} - 3\mathrm{Bo})^2 \times \mathbb{P}_{10}$

Phase space analysis: global behaviour

Contains ≈ 11 cells with 0 to 3 real roots such as $y > 1$



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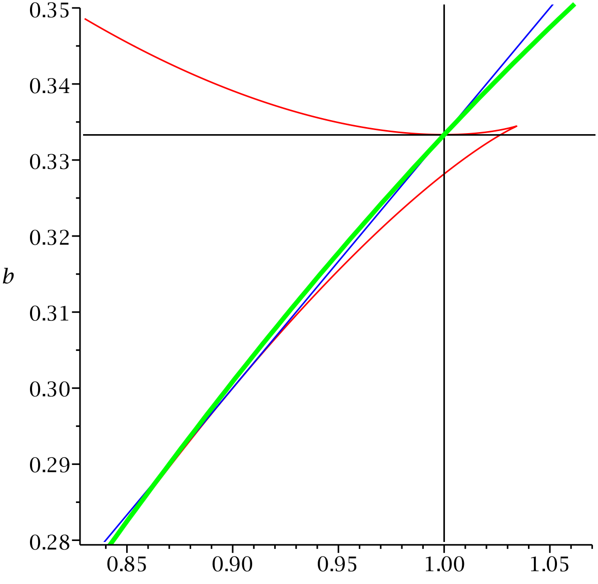
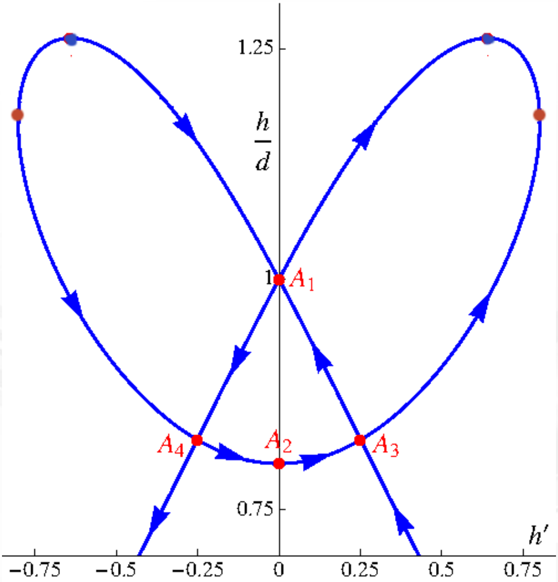


Phase-space analysis: weakly singular solitary wave

A particular example for $\mathrm{Fr} = 0.8$, $\mathrm{Bo} = 0.3538557 > 1/3$



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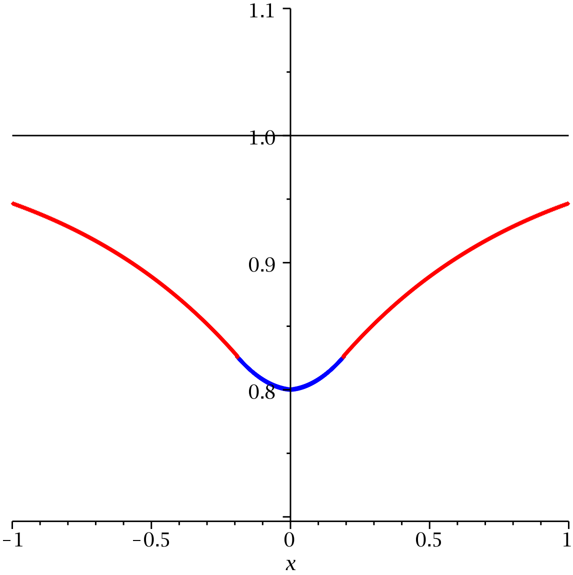


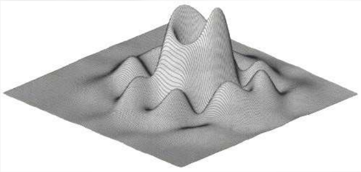
Conclusions & Perspectives

Conclusions:

- Capillary-gravity solitary waves were analyzed in shallow water regime
- Fully nonlinear & weakly dispersive model
  - Phase-space analysis using the methods of the algebraic geometry
- Only two types of regular solitary waves ($+a$, $-a$)

Perspectives:

- Analysis of periodic CG-waves
- Go to 3D !
  - Compute fully nonlinear lump-solitary waves



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http://www.Denys-Dutykh.com/

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