Visco potential free-surface flows

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Importance of viscous effects

Experimental evidences

1. J. Bona, W. Pritchard & L. Scott, *An Evaluation of a Model Equation for Water Waves.* Phil. Trans. R. Soc. Lond. A, 1981, 302, 457–510

In « Resumé » section :

> *[...] it was found that the inclusion of a dissipative term was much more important than the inclusion of the nonlinear term, although the inclusion of the nonlinear term was undoubtedly beneficial in describing the observations [...]*

2. Boussinesq (1895), Lamb (1932) derived a formula

$$\frac{d\alpha}{dt} = -2\nu k^2 \alpha(t)$$

Mechanisms of dissipation

1. Wave breaking
   - The main effect of wave breaking is the dissipation of energy
   - $\Rightarrow$ This can be modelled by adding dissipative terms in coastal regions where the wave becomes steeper
     Ref. : [Melville], [Popinet]
2. Turbulence
   - For tsunami wave $Re \geq 10^6$, so the flow is turbulent
   - $\Rightarrow$ energy extraction from waves in upper ocean
     Ref. : [Thorpe]
3. Boundary layers
   - Regions where the viscosity is the most important
     1. free surface boundary layer (ref. : [Hammerton & Bassom])
     2. bottom boundary layer
4. Molecular viscosity
   - The least important factor for long waves

Energy balance in a fluid flow

- We assume that flow is governed by incompressible Navier-Stokes equations :

$$\begin{aligned} \nabla \cdot \vec{u} &= 0 \\ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} &= \vec{g} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \tau \end{aligned}$$

- We multiply the second equation by $\vec{u}$ and integrate on control volume Ω :

$$\frac{1}{2}\int_{\Omega} \frac{\partial}{\partial t}\left(\rho|\vec{u}|^2\right) \, d\Omega + \frac{1}{2}\int_{\partial\Omega} \rho|\vec{u}|^2\vec{u} \cdot \vec{n} \, d\sigma =$$

$$= \int_{\partial\Omega} (-p\mathbb{I} + \tau)\vec{n} \cdot \vec{u} \, d\sigma + \int_{\Omega} \rho\vec{g} \cdot \vec{u} \, d\Omega - \underbrace{\frac{1}{2\mu}\int_{\Omega} \tau : \tau \, d\Omega}_{\mathcal{T}}$$

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Anatomy of dissipation

Estimation of viscous dissipation rate

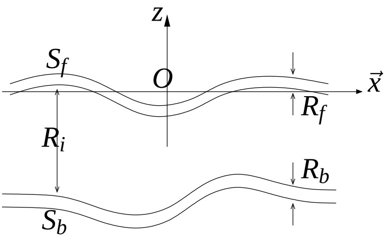

FIG.: Flow regions

- Interior region :
  $\mathcal{T}_{R_i} \sim \frac{1}{\mu}\left(\mu \frac{a}{t_0 \ell}\right)^2 \cdot \ell^3 \sim \mu$
- Free surface boundary layer :
  $\mathcal{T}_{R_f} \sim \frac{1}{\mu}\left(\mu \frac{a}{t_0 \ell}\right)^2 \cdot \delta \ell^2 \sim \mu^{\frac{3}{2}}$
- Bottom boundary layer :
  $\mathcal{T}_{R_b} \sim \frac{1}{\mu}\left(\mu \frac{a}{t_0 \delta}\right)^2 \cdot \delta \ell^2 \sim \mu^{\frac{1}{2}}$

The previous scalings suggest us the following diagram :

$$\underbrace{\mathcal{O}(\mu^{\frac{1}{2}})}_{R_b} \hookrightarrow \underbrace{\mathcal{O}(\mu)}_{R_i \bigcup S_f} \hookrightarrow \underbrace{\mathcal{O}(\mu^{\frac{3}{2}})}_{R_f} \hookrightarrow \underbrace{\mathcal{O}(\mu^2)}_{S_f} \hookrightarrow \ldots$$

Visco potential flows

How to add dissipation in potential flows ?

1. Consider Navier-Stokes equations
2. Write Helmholtz-Leray decomposition $\vec{u} = \nabla\phi + \nabla \times \vec{\psi}$
3. Express vortical components $\vec{\psi}$ of velocity field in terms of potential ϕ and η using (pseudo) differential (fractional) operators

Kinematic condition :

$$\eta_t = \phi_z + \psi_{2x} - \psi_{1y} \quad \Rightarrow \quad \eta_t = \phi_z + 2\nu\nabla^2\eta$$

Dynamic condition :

$$\phi_t + g\eta + 2\nu\phi_{zz} + \mathcal{O}(\nu^{\frac{3}{2}}) = 0$$

D. Dutykh, F. Dias. *Viscous potential free-surface flows in a fluid layer of finite depth*. CRAS, Ser. I, 345 (2007), 113–118

Boundary layer correction

Bottom boundary condition

1. Consider semi-infinite domain : $z > -h$
2. Use pure Leray decomposition : $\vec{v} = \nabla\phi + \vec{u}, \quad \nabla \cdot \vec{u} = 0$
3. Introduce boundary layer coordinate : $\zeta = \frac{(z+h)}{\delta}$, where $\delta = \sqrt{\nu}$
4. Asymptotic expansion : $\phi = \phi_0 + \delta\phi_1 + \dots$

Bottom boundary condition :

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = -\sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\phi_{zz}|_{z=-h}}{\sqrt{t-\tau}}\, d\tau = -\sqrt{\nu}\mathcal{I}[\phi_{zz}]$$

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Nonlocal visco potential formulation

Resulting governing equations

- Continuity equation

$$\Delta\phi = 0, \quad (x,y,z) \in \Omega,$$

- Kinematic free surface condition

$$\frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta = \frac{\partial\phi}{\partial z} + 2\nu\nabla^2\eta, \quad z = \eta(x,y,t),$$

- Dynamic free surface condition

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 2\nu\nabla^2\phi, \quad z = \eta(x,y,t).$$

- Kinematic bottom condition

$$\frac{\partial\phi}{\partial z} + \nabla\phi\cdot\nabla h = -\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{\phi_{zz}}{\sqrt{t-\tau}}\,d\tau, \quad z = -h(x,y),$$

Linear dispersion relation

Novel visco potential formulation

We look for the following periodic solutions :

$\phi(\vec{x}, z, t) = \varphi(z)\ e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, $\eta(\vec{x}, t) = \eta_0\ e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ with $h \equiv const$

- If we do not modify bottom boundary condition :

$$\omega(|\vec{k}|) = \sqrt{g|\vec{k}|\tanh(|\vec{k}|h)} - 2i\nu|\vec{k}|^2$$

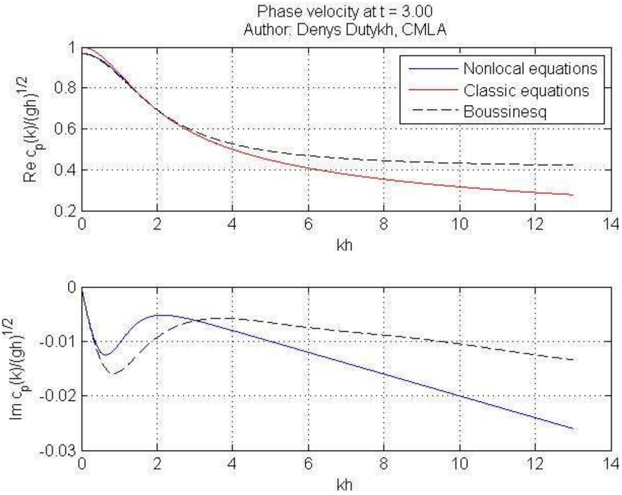
- Dispersion relation with nonlocal term :

$$(i\omega - 2\nu|\vec{k}|^2)^2 + g|\vec{k}|\tanh(|\vec{k}|h) = |\vec{k}|\sqrt{\frac{i\nu}{\omega}}\Big((i\omega - 2\nu|\vec{k}|^2)\tanh(|\vec{k}|h) + g|\vec{k}|\Big)\operatorname{erf}(\sqrt{-i\omega t})$$

  - Time dependent dispersion : $\omega = \omega(|\vec{k}|, t)$

Dependence of phase velocity on time

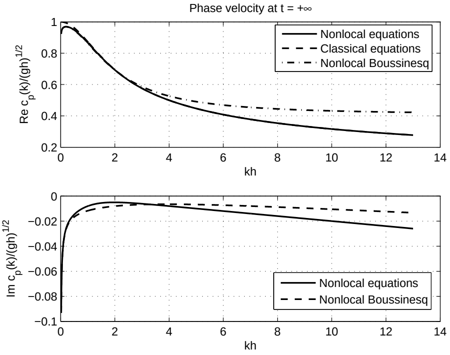
With nonlocal term : $\omega = \omega(|\vec{k}|, t)$



Phase velocity at infinite time

We take analytic limit as $t \to +\infty$

$$D(\omega_\infty, k) := (i\omega_\infty - 2\nu k^2)^2 + gk\tanh(kh) - \sqrt{\frac{\nu}{\omega_\infty}}ke^{i\frac{\pi}{4}}\Big((i\omega_\infty - 2\nu k^2)^2\tanh(kh) + gk\Big) \equiv 0$$



Long wave approximation

1. Nonlocal Boussinesq equations :
   - Mass conservation :

$$\eta_t + \nabla\cdot((h+\eta)\vec{u}) + A_\theta h^3 \nabla^2(\nabla\cdot\vec{u}) = 2\nu\Delta\eta + \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\nabla\cdot\vec{u}}{\sqrt{t-\tau}}\, d\tau$$

   - Horizontal momentum :

$$\vec{u}_t + \frac{1}{2}\nabla|\vec{u}|^2 + g\nabla\eta - B_\theta h^2 \nabla(\nabla\cdot\vec{u}_t) = 2\nu\Delta\vec{u}$$

2. Nonlocal KdV equation :

$$\eta_t + \sqrt{\frac{g}{h}}\Big((h + \frac{3}{2}\eta)\eta_x + \frac{1}{6}h^3\eta_{xxx} - \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\eta_x}{\sqrt{t-\tau}}\, d\tau\Big) = 2\nu\eta_{xx}$$

Solitary wave attenuation

Effect of nonlocal term on the amplitude

Numerical solution to nonlocal Boussinesq equations

- Solitary wave initial condition
- Fourier-type spectral method
- Comparison between :
  1. Classical Boussinesq equations
  2. Local dissipative terms
  3. Local + nonlocal dissipation

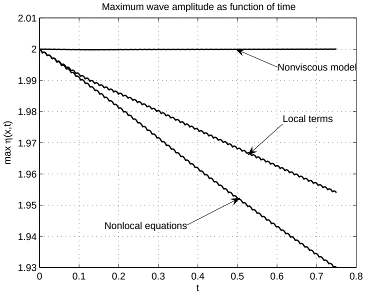


FIG.: Soliton amplitude

Solitary wave attenuation
Effect of nonlocal term on the amplitude

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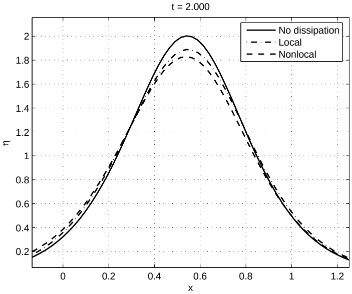

FIG.: Zoom on soliton crest

- Consider nonlocal dissipative Airy equation

$$\eta_t + \sqrt{\frac{g}{h}}\left(h\eta_x + \frac{1}{6}h^3\eta_{xxx} - \sqrt{\frac{\nu}{\pi}}\int_0^t \frac{\eta_x}{\sqrt{t-\tau}}\,d\tau\right) = 2\nu\eta_{xx}$$

- Special form of solutions

$$\eta(x,t) = \mathcal{A}(t)e^{ik\xi}, \quad \xi = x - \sqrt{gh}t, \quad \mathcal{A}(t) \in \mathbb{C}$$

Integro-differential equation :

$$\frac{d|\mathcal{A}|^2}{dt} + 4\nu k^2|\mathcal{A}(t)|^2 + ik\sqrt{\frac{g\nu}{\pi h}}\int_0^t \frac{\bar{\mathcal{A}}(t)\mathcal{A}(\tau) - \mathcal{A}(t)\bar{\mathcal{A}}(\tau)}{\sqrt{t-\tau}}\,d\tau = 0$$

Conclusions and perspectives

Conclusions :

- Novel visco potential formulation was developed
  1. Takes into account dissipation
  2. Simpler than Navier-Stokes equations
- Dispersion relation was analysed
- Corresponding long wave models were derived

Perspectives :

- Include other mechanisms of dissipation
- Efficient computation of nonlocal term

Thank you for your attention !

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http://www.cmla.ens-cachan.fr/~dutykh
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