Fully nonlinear weakly dispersive travelling capillary–gravity waves

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Université de Lyon 1, August 28, 2014

Acknowledgements

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  Université de Nice Sophia Antipolis
- André GALLIGO: Emiritus Professor
  Université de Nice Sophia Antipolis

Outline

1. Shallow capillary–gravity waves
   - Travelling waves
   - Phase-space analysis

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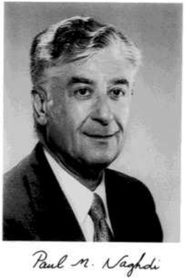
<u>Credits</u>:

- Lord RAYLEIGH (1876) [1]
- F. SERRE (1953) [2]
- C. SU & C. GARDNER (1969) [3]
- A. GREEN & P. NAGHDI (1976) [4]
- E. PELINOVSKY & ZHELEZNYAK (1985) [5]

Modern derivations:

- Asymptotic methods: Lannes & Bonneton (2009) [6]
- Variational methods:
  - Particle description: Miles & Salmon (1985) [7]
  - Eulerian description: Clamond & DD (2012) [8]



- Governing equations (the mass conservation + momentum):

$$h_t + [h\,\bar{u}]_x = 0,$$

$$\bar{u}_t + \bar{u}\,\bar{u}_x + g\,h_x + \tfrac{1}{3}h^{-1}\partial_x\Big[h^2\,\tilde{\gamma}\Big] = \tau\Big[h_x\left(1+h_x^2\right)^{-1/2}\Big]_{xx}$$

- Vertical acceleration:

$$\tilde{\gamma} = h\,(\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}) = 2\,h\,\bar{u}_x^2 - h\,[\bar{u}_t + \bar{u}\,\bar{u}_x]_x$$

- Conservative form:

$$[h\,\bar{u}]_t + \Big[h\,\bar{u}^2 + \tfrac{1}{2}g\,h^2 + \tfrac{1}{3}h^2\,\tilde{\gamma} - \tau\,R\Big]_x = 0$$

- Surface tension:

$$R = h\,h_{xx}\left(1+h_x^2\right)^{-3/2} + \left(1+h_x^2\right)^{-1/2},$$

Serre's equations: the conservation laws

- Momentum conservations:

$$\Big[h\bar{u} - \tfrac{1}{3}(h^3\bar{u}_x)_x\Big]_t + \Big[h\bar{u}^2 + \tfrac{1}{2}gh^2 - \tfrac{1}{3}2h^3\bar{u}_x^2 - \tfrac{1}{3}h^3\bar{u}\bar{u}_{xx} - h^2h_x\bar{u}\bar{u}_x - R\Big]_x = 0$$

- Tangential velocity at the free surface:

$$\Big[\bar{u} - \frac{(h^3\bar{u}_x)_x}{3\,h}\Big]_t + \Big[\tfrac{1}{2}\bar{u}^2 + gh - \tfrac{1}{2}h^2\,\bar{u}_x^2 - \frac{\bar{u}\,(h^3\bar{u}_x)_x}{3\,h} - \frac{\tau\,h_{xx}}{(1+h_x^2)^{3/2}}\Big]_x = 0$$

- Energy conservation:

$$\Big[\tfrac{1}{2}h\,\bar{u}^2 + \tfrac{1}{6}h^3\,\bar{u}_x^2 + \tfrac{1}{2}g\,h^2 + \tau\sqrt{1+h_x^2}\Big]_t +$$

$$\Big[\big(\tfrac{1}{2}\bar{u}^2 + \tfrac{1}{6}h^2\,\bar{u}_x^2 + gh + \tfrac{1}{3}h\gamma - \frac{\tau R}{h}\big)h\,\bar{u} + \tau\,\bar{u}\sqrt{1+h_x^2} + \frac{\tau\,h\,h_x\,\bar{u}_x}{\sqrt{1+h_x^2}}\Big]_x = 0$$

Serre–CG equations: travelling waves

$\mathrm{Fr} = c^2/gd$, $\mathrm{Bo} = \tau/gd^2$, $\mathrm{We} = \mathrm{Bo}\,/\,\mathrm{Fr} = \tau/c^2 d$

- Mass conservation: $\bar{u} = -cd\ /\ h$, $d =< h >= \frac{1}{2\ell}\int_{-\ell}^{\ell} h\,\mathrm{d}x$
- Momentum conservations lead:

$$\frac{\mathrm{Fr}\,d}{h} + \frac{h^2}{2\,d^2} + \frac{\tilde{\gamma}h^2}{3gd^2} - \frac{\mathrm{Bo}\,hh_{xx}}{\left(1+h_x^2\right)^{\frac{3}{2}}} - \frac{\mathrm{Bo}}{\left(1+h_x^2\right)^{\frac{1}{2}}} = \mathrm{Fr} + \frac{1}{2} - \mathrm{Bo} + K_1$$

- Tangential velocity:

$$\frac{\mathrm{Fr}\ d^2}{2\,h^2} + \frac{h}{d} + \frac{\mathrm{Fr}\ d^2\,h_{xx}}{3\,h} - \frac{\mathrm{Fr}\ d^2\,h_x^2}{6\,h^2} - \frac{\mathrm{Bo}\ d\,h_{xx}}{\left(1+h_x^2\right)^{\frac{3}{2}}} = \frac{\mathrm{Fr}}{2} + 1 + \frac{\mathrm{Fr}\ K_2}{2}$$

$$\tilde{\gamma}/g \;=\; \mathrm{Fr}\,d^3 h_{xx}/h^2 \;-\; \mathrm{Fr}\,d^3 h_x^2/h^3$$

- Integration constants:

$$K_2 \;=\; \left\langle\, \frac{(3+h_x^2)\,d^2}{3\,h^2} \;-\; 1 \,\right\rangle$$

$$K_1 \;+\; \tfrac{1}{2} \;+\; \mathrm{Fr} \;-\; \mathrm{Bo} \;=\; \left\langle\, \frac{\mathrm{Fr}\ d^2}{h^2} + \frac{1}{2} - \frac{\mathrm{Bo}\ d\ /\ h}{\left(1+h_x^2\right)^{\frac{1}{2}}} \,\right\rangle \Big/ \left\langle\, \frac{d}{h} \,\right\rangle.$$

- Combination of two equations:

$$\frac{\mathrm{Fr}\,d}{2\,h}-\frac{h^2}{2\,d^2}-\frac{\mathrm{Fr}\,d\,h_x^2}{6\,h}-\frac{\mathrm{Bo}}{(1+h_x^2)^{\frac{1}{2}}}+\frac{(\mathrm{Fr}+2+\mathrm{Fr}\,K_2)\,h}{2\,d}=\mathrm{Cnst}$$

- Consider solitary waves ($K_1 = K_2 \equiv 0$):

$$F(h,h') \equiv \frac{\mathrm{Fr}\,h'^2}{3} + \frac{2\,\mathrm{Bo}\,h/d}{\left(1+h'^2\right)^{\frac{1}{2}}} - \mathrm{Fr} + \frac{(2\mathrm{Fr}+1-2\mathrm{Bo})\,h}{d} - \frac{(\mathrm{Fr}+2)\,h^2}{d^2} + \frac{h^3}{d^3} = 0$$

- Property: $h(x) \equiv h(-x)$
- At the crest of a regular wave: $h(0) = d + a$, $h'(0) = 0$

$$\mathrm{Fr} = 1 + a/d$$

- Combination of two equations:

$$\frac{\mathrm{Fr}\, d}{2\,h} - \frac{h^2}{2\,d^2} - \frac{\mathrm{Fr}\, d\, h_x^2}{6\,h} - \frac{\mathrm{Bo}}{\left(1 + h_x^2\right)^{\frac{1}{2}}} + \frac{(\mathrm{Fr} + 2 + \mathrm{Fr}\, K_2)\, h}{2\,d} = \mathrm{Cnst}$$

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- We are interested in solutions: $h(\infty) = d$, $h'(\infty) = 0$

- Pure gravity waves: $\mathrm{Bo} \to 0$

$$h = d + a\,\mathrm{sech}^2(\kappa x/2), \quad (\kappa d)^2 = 3\,a\,/\,(d+a), \quad \mathrm{Fr} = 1 + a\,/\,d$$

- Pure capillary waves: $\mathrm{Fr} \to \infty$, $\mathrm{Bo} \to \infty$, $\mathrm{We} = \mathrm{Const}$

$$\frac{h'^2}{3} + \frac{2\,\mathrm{We}\;h/d}{\left(1 + h'^2\right)^{\frac{1}{2}}} - 1 + \frac{2\,(1 - \mathrm{We}\,)\,h}{d} - \frac{h^2}{d^2} = 0$$

In the limit $\mathrm{We} \to 0$:

- Capillary wave equation reads: $h' = \pm\sqrt{3}\,(1 - h/d)$
- Solitary wave with angular crest:

$$h = d + a\,\exp\left(-\sqrt{3}\,|x|\right)$$

- Pure gravity waves: $\mathrm{Bo} \to 0$

$$h = d + a\,\mathrm{sech}^2(\kappa x/2), \quad (\kappa d)^2 = 3\,a\,/\,(d+a), \quad \mathrm{Fr} = 1 + a\,/\,d$$

- Pure capillary waves: $\mathrm{Fr} \to \infty$, $\mathrm{Bo} \to \infty$, $\mathrm{We} = \mathrm{Const}$

$$\frac{h'^2}{3} + \frac{2\,\mathrm{We}\,h/d}{\left(1+h'^2\right)^{\frac{1}{2}}} - 1 + \frac{2\,(1-\mathrm{We}\,)\,h}{d} - \frac{h^2}{d^2} = 0$$

Droplet solution (opposite limit $\mathrm{Fr} \to 0$):

$$(h/d)\left[2\mathrm{Bo}\,(1+h'^2)^{-\frac{1}{2}} + 1 - 2\mathrm{Bo} - 2h/d + (h/d)^2\right] = 0$$

$$\left[1 - \frac{1}{(1+\eta'^2)^{\frac{1}{2}}}\right]^{1/2} = \pm\frac{\eta/d}{\sqrt{2\mathrm{Bo}}}, \quad \eta = h - d$$

Serre–CG equation: Limiting cases II

Introduce a simplification to equations

- Small slope approximation [9]:

$$\left(1 + h'^2\right)^{-\frac{1}{2}} \approx 1 - \tfrac{1}{2} h'^2 + \ldots$$

- Master equation becomes:

$$\Big(\frac{\mathrm{Fr}^2}{3} - \frac{\mathrm{Bo}\, h}{d}\Big) h'^2 = \mathrm{Fr}^2 - \frac{(2\mathrm{Fr}^2 + 1)\, h}{d} + \frac{(\mathrm{Fr}^2 + 2)\, h^2}{d^2} - \frac{h^3}{d^3}$$

- Change of independent variables:

$$\mathrm{d}\xi = \left|1 - 3\,\mathrm{We}\, h / d\right|^{-\frac{1}{2}} \mathrm{d}x \qquad (\star)$$

Analytical solution ($\equiv$ pure gravity case):

- $h(\xi) = d + a\, \mathrm{sech}^2(\kappa\xi/2)$
- $(\kappa d)^2 = 3a/(d+a), \ \mathrm{Fr} = 1 + a/d$
- $x(\xi)$ can be found from $(\star)$

- Exponentially decaying solitary waves:

$$h(x) \sim d + a \exp(-\kappa x), \quad x \to +\infty, \quad \kappa > 0$$

'Dispersion' relation:

$$\mathrm{Fr}^2 = \frac{3 - 3\,\mathrm{Bo}\,(\kappa d)^2}{3 - (\kappa d)^2} \quad \text{or} \quad (\kappa d)^2 (\mathrm{Fr} - 3\,\mathrm{Bo}) = 3\left(\mathrm{Fr}^2 - 1\right)$$

- κ can be ony real **or** purely imaginary
  - Solitary waves **or** Periodic waves
- Critical values:
  - $\mathrm{Fr} = 1$, $\mathrm{Bo} = 1/3$, $\kappa d = \sqrt{3}$ or $\mathrm{Bo} = \frac{1}{3}\mathrm{Fr}$

- Algebraic decay:

$$h(x) \sim d + a(\kappa x)^{-\alpha}, \quad x \to +\infty, \quad \alpha > 1$$

- Then necessarily $\Longrightarrow \quad \mathrm{Fr} \equiv 1$
- $\alpha = 1 \Longrightarrow \mathrm{Bo} \neq \frac{1}{3}, \qquad \alpha > 2 \Longrightarrow \mathrm{Bo} = \frac{1}{3}$

- Two-parameter $(\mathrm{Fr}, \mathrm{Bo})$ family of real algebraic curves in $\mathbb{R}^2$:

$$F_{\mathrm{Fr},\mathrm{Bo}}(h,k) := \frac{\mathrm{Fr}}{3}k^2 + 2\frac{\mathrm{Bo}\,h}{(1+k^2)^{\frac{1}{2}}} + \\ - \mathrm{Bo} + (2\mathrm{Fr} - 2\mathrm{Bo} + 1)h - (\mathrm{Fr} + 2)h^2 + h^3 = 0$$

- With asymptotic behaviour at $x \to \infty$ ($k := h'$):

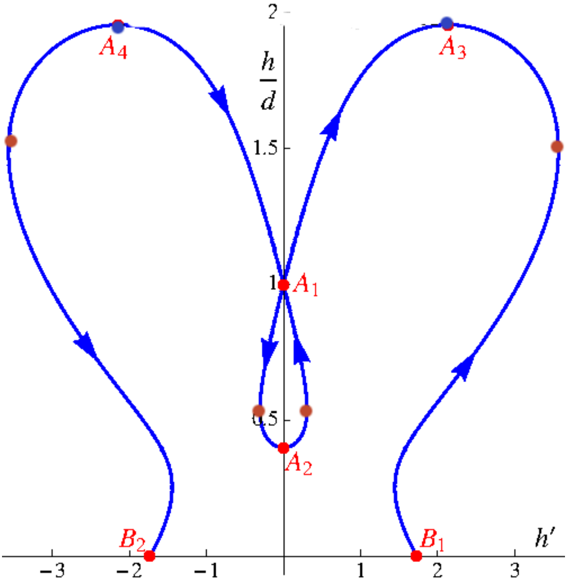
$$h(\infty) = 1, \quad k(\infty) = 0$$

Typical workflow with a parametrized curve:

- Find multiple points (with horizontal tangent): $\partial_k F_{\mathrm{Fr},\mathrm{Bo}} = 0$
- Find points with vertical tangent: $\partial_h F_{\mathrm{Fr},\mathrm{Bo}} = 0$
- Decompose it into oriented branches ($k \gtrless 0$, $h \nearrow\searrow 0$)
- Study the family of algebraic curves $F_{\mathrm{Fr},\mathrm{Bo}}(h,k) \in \mathbb{R}^2$ with certified topology methods [11]

Phase-space analysis: depression wave

A particular example for $\mathrm{Fr} = 0.4$, $\mathrm{Bo} = 0.9 > 1/3$



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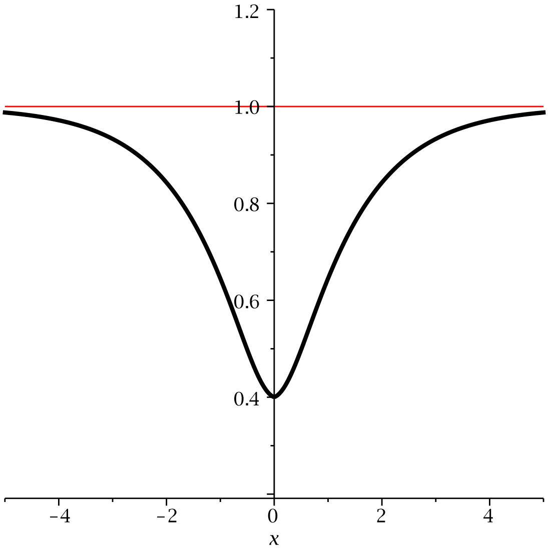
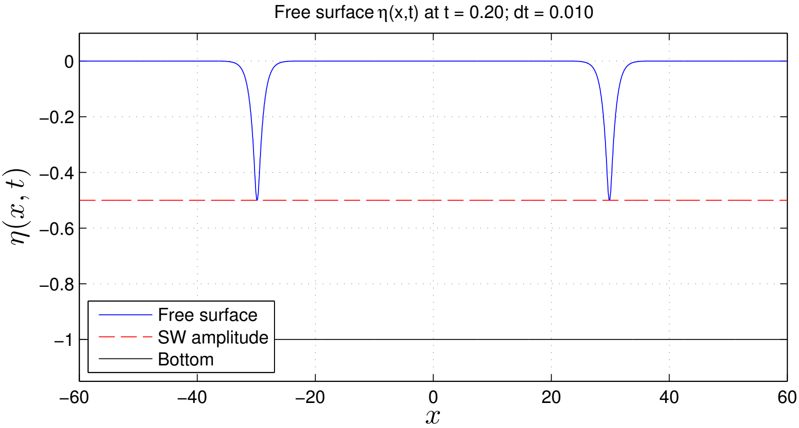


Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\mathrm{Fr} = 1/2$, $\mathrm{Bo} = 1/2$



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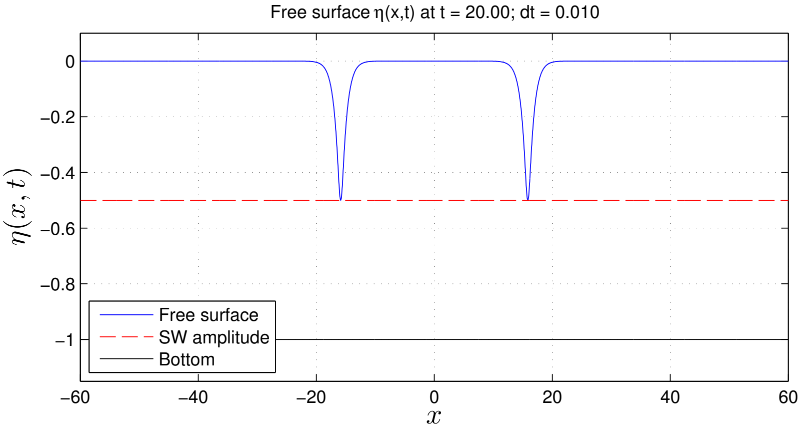

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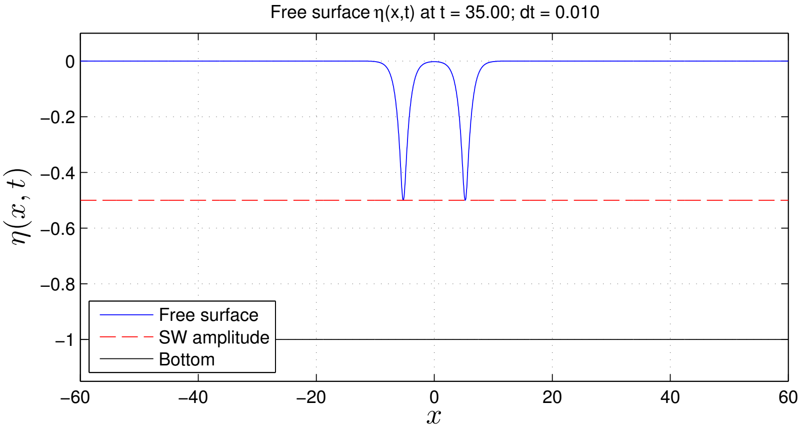
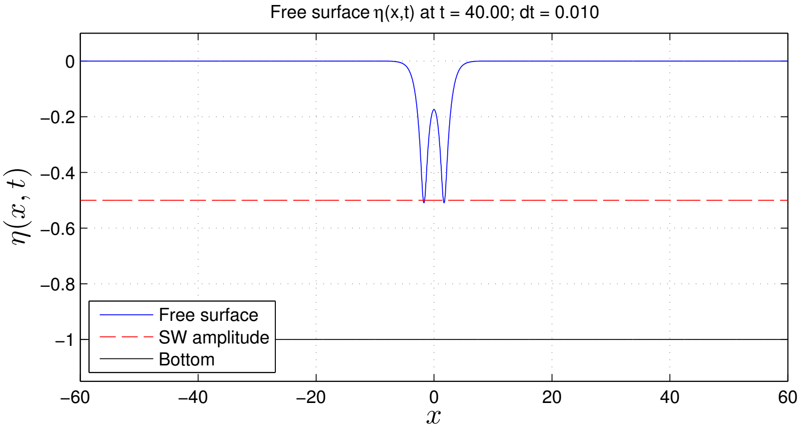
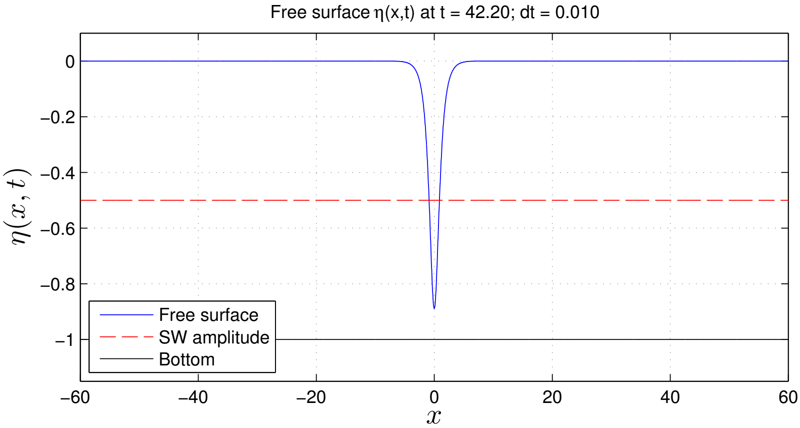
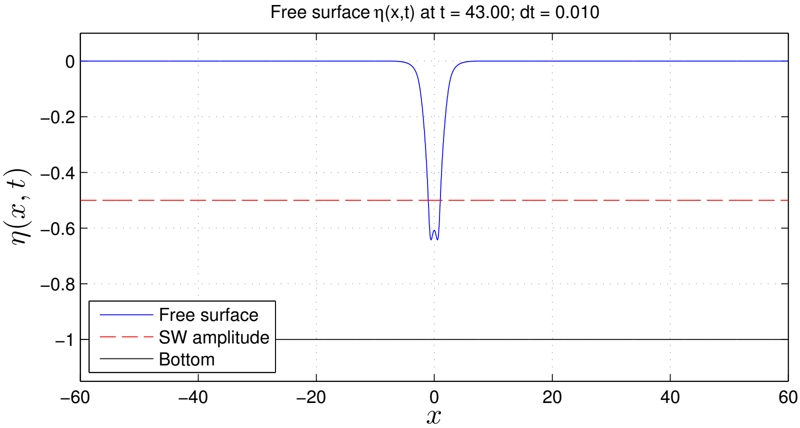
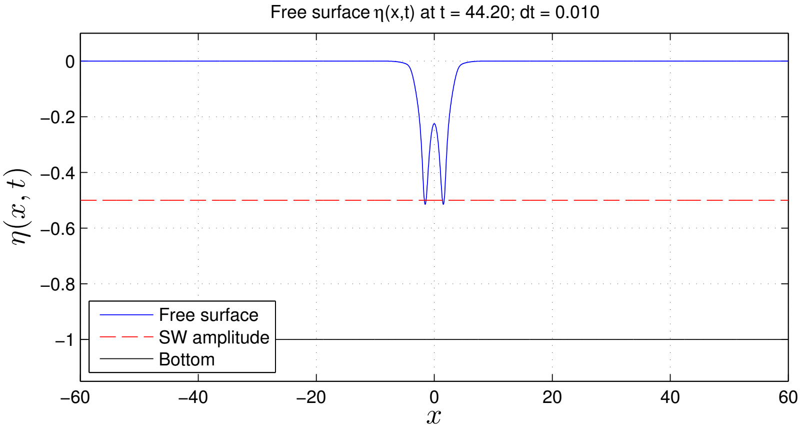



Free surface η(x,t) at t = 40.00; dt = 0.010
0
–0.2
–0.4
–0.6
–0.8
–1
η(x, t)
Free surface
SW amplitude
Bottom
–60
–40
–20
0
20
40
60
x



Free surface η(x,t) at t = 42.20; dt = 0.010
0
–0.2
–0.4
–0.6
–0.8
–1
$\eta(x,t)$
–60
–40
–20
0
20
40
60
x
Free surface
SW amplitude
Bottom


Free surface η(x,t) at t = 43.00; dt = 0.010
0
–0.2
–0.4
–0.6
–0.8
–1
η(x, t)
–60
–40
–20
0
20
40
60
x
Free surface
SW amplitude
Bottom


Free surface η(x,t) at t = 44.20; dt = 0.010
$\eta(x,t)$
0
–0.2
–0.4
–0.6
–0.8
–1
–60
–40
–20
0
20
40
60
x
Free surface
SW amplitude
Bottom

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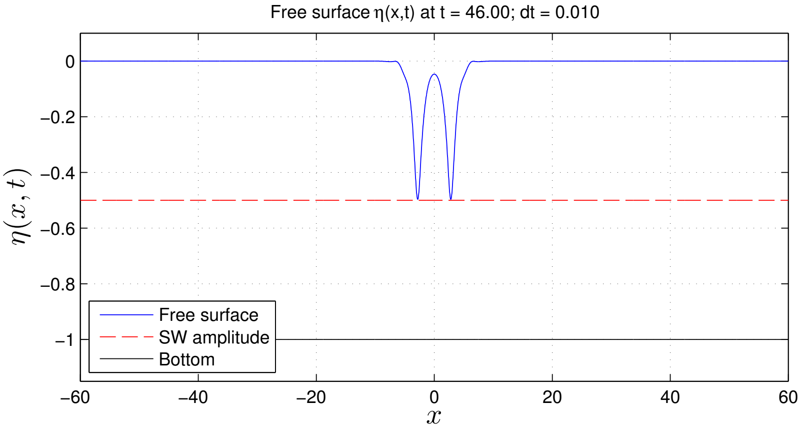

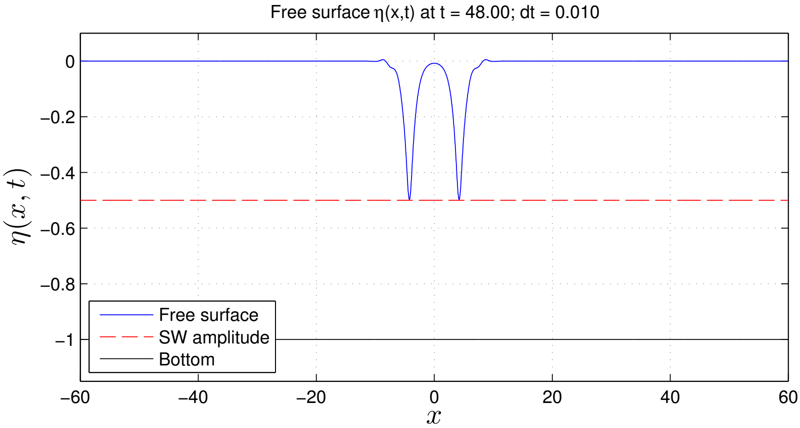
Free surface η(x,t) at t = 48.00; dt = 0.010
0
–0.2
–0.4
–0.6
–0.8
–1
$\eta(x,t)$
–60
–40
–20
0
20
40
60
x
Free surface
SW amplitude
Bottom

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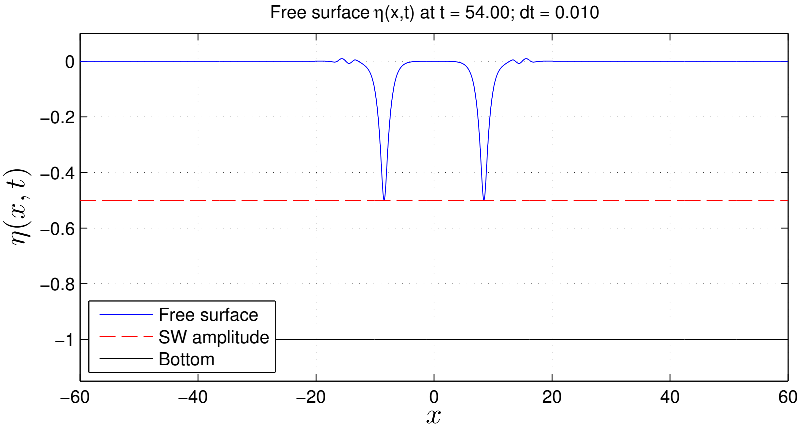
Solitary waves of depression: $a/d = -1/2$, $\mathrm{Fr} = 1/2$, $\mathrm{Bo} = 1/2$



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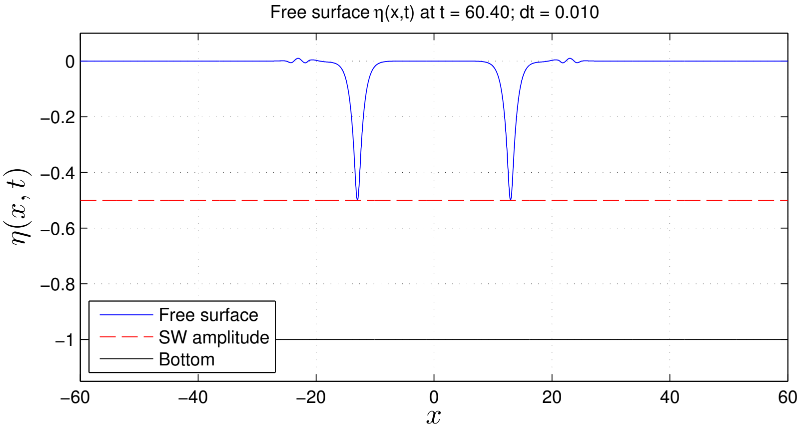
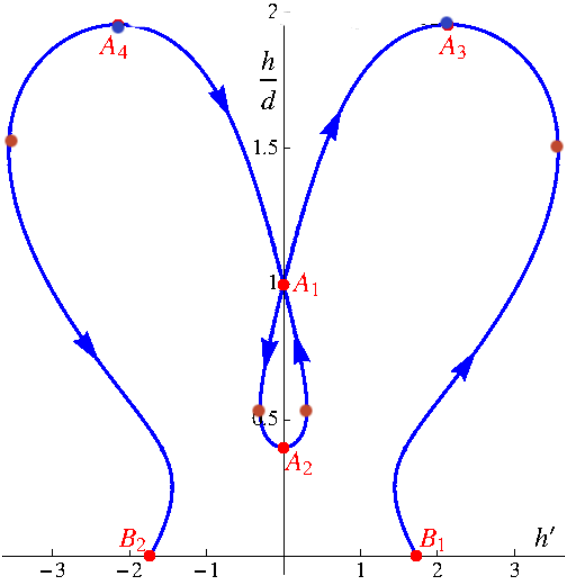


Phase-space analysis: peakon of depression

A particular example for $\mathrm{Fr} = 0.4$, $\mathrm{Bo} = 0.9 > 1/3$



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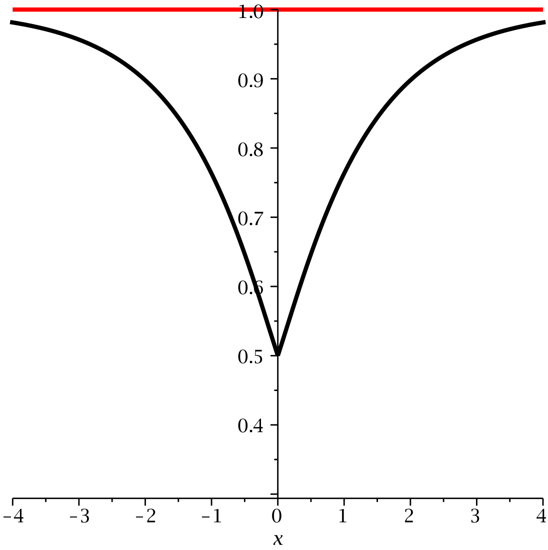

Phase-space analysis: multi-peakon of depression

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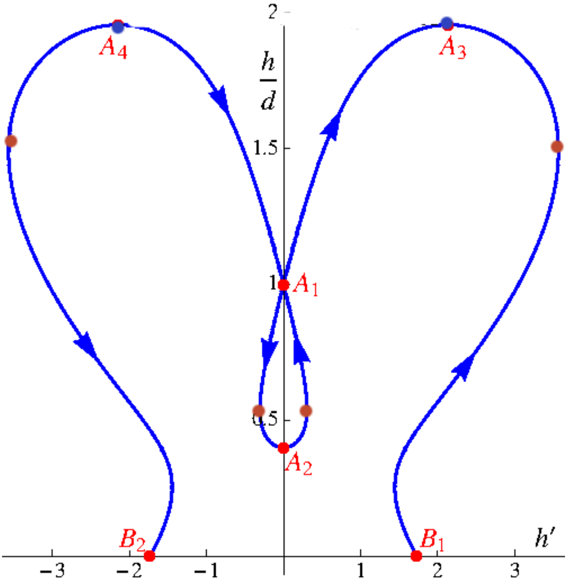

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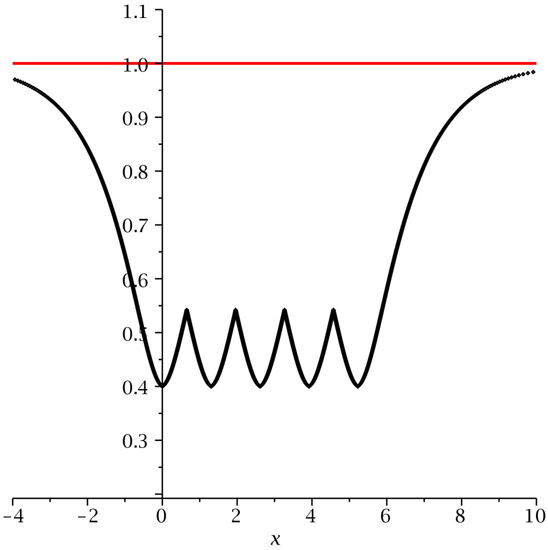


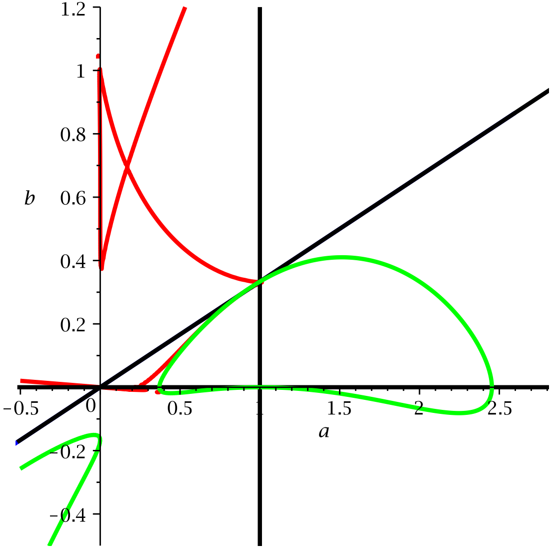
- Points with horizontal tangent satisfy:
  - $F_{\mathrm{Fr}\,,\,\mathrm{Bo}}(k,h) = 0$
  - $\partial_k F_{\mathrm{Fr}\,,\,\mathrm{Bo}}(k,h) = 0$
- The 2nd equation can be solved analytically:
  - $k = 0$
  - $\mathrm{Fr}\,(k^2+1)^3 = 9\mathrm{Bo}^2 h^2$
- To avoid cubic roots, change of variables:
  - $k^2 = y^2 - 1$, $y \geq 1$
  - Wave height can be expressed as $h = \frac{\mathrm{Fr}}{3\mathrm{Bo}} Y^3$
- Polynomial equation in y:

$$f := \mathrm{Fr}^2 y^9 - (3\mathrm{Fr} - 2)\mathrm{Fr}\,\mathrm{Bo}\, y^6 + 9\mathrm{Bo}^2(1 + 2\mathrm{Fr} - 2\mathrm{Bo})y^3 + 27\mathrm{Bo}^3 y^2 - 36\mathrm{Bo}^3$$

- Adapted tool to describe the real roots in $(\mathrm{Fr}\,,\ \mathrm{Bo})$ space: discriminant locus! $\mathcal{D} = (\mathrm{Fr} - 3\mathrm{Bo})^2 \times \mathbb{P}_{10}$

Phase space analysis: global behaviour

Contains $\approx$ 11 cells with 0 to 3 real roots such as $y > 1$



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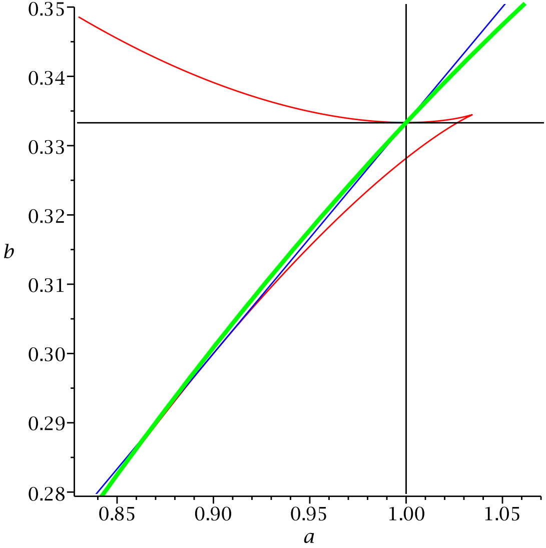
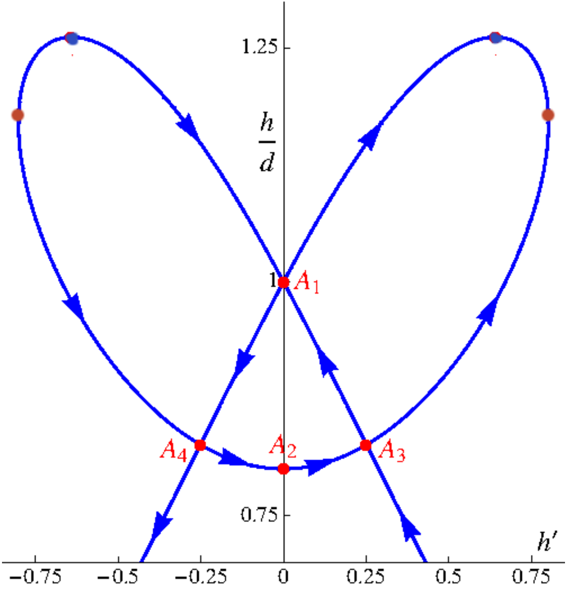


Phase-space analysis: weakly singular solitary wave

A particular example for $\mathrm{Fr} = 0.8$, $\mathrm{Bo} = 0.3538557 > 1/3$



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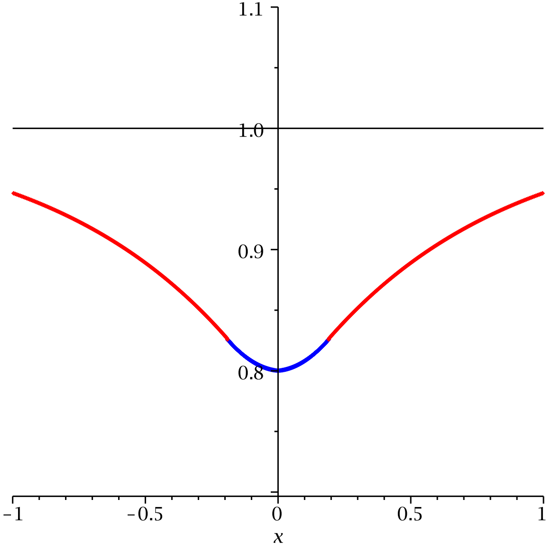
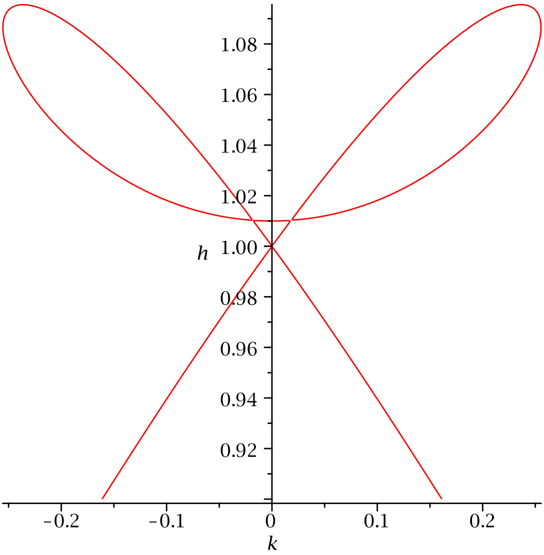


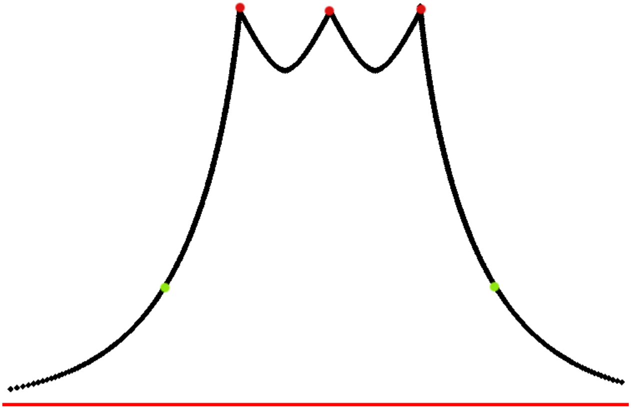
Phase-space analysis: weakly singular solitary wave

A particular example for $\mathrm{Fr} = 1.01$, $\mathrm{Bo} = 0.333412 > 1/3$



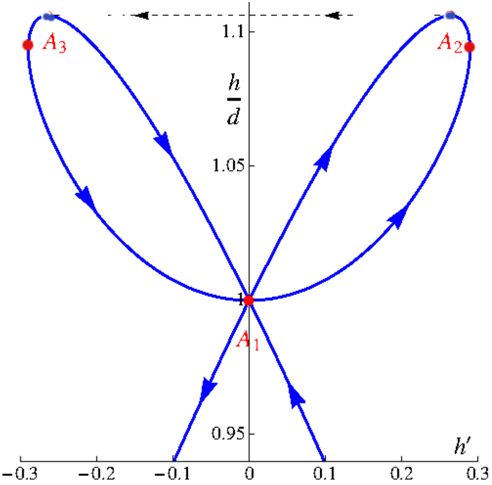
Phase-space analysis: weakly singular solitary wave

A particular example for $\mathrm{Fr} = 1.01$, $\mathrm{Bo} = 0.333412 > 1/3$



Phase-space analysis: wave with algebraic decay

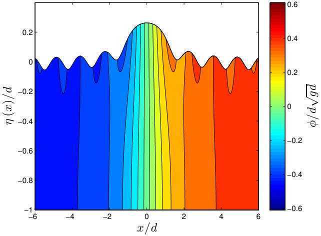
A particular example for $\mathrm{Fr} = 1.0$, $\mathrm{Bo} = 1/3$



Conclusions & Perspectives

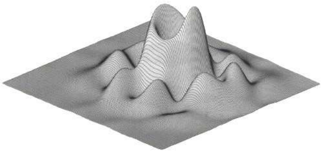
Conclusions:

- Capillary–gravity solitary waves were analyzed in shallow water regime
- Fully nonlinear & weakly dispersive model
  - Algebraic geometry +
  - Certified topology methods
- Only two types of regular solitary waves ($+a$, $-a$)



Perspectives:

- Go to 3D !
  - Compute fully nonlinear lump-solitary waves



Thank you for your attention!

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