Clamond & Grue spectral scheme

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Short Course on "*Modeling of Nonlinear Ocean Waves*"

THE CREATORS

AND ALSO DORIAN FRUCTUS (OUT OF ACADEMIA NOW)...

(a) Didier

(b) John

The creators

and also Dorian Fructus (out of academia now)...

Figure: Dorian

Some history of the method

- Pioneering work for 2D waves in deep water (with some hints about extending to 3D) [CG01]:
  - Clamond, D., & Grue, J. (2001). *A fast method for fully nonlinear water-wave computations*. J. Fluid. Mech., **447**, 337-355.
- Full extension to 3D [FCKG05]:
  - Fructus, D., Clamond, D., Kristiansen, O., & Grue, J. (2005). *An efficient model for threedimensional surface wave simulations. Part I: Free space problems*. J. Comput. Phys., **205**, 665-685.
- Generation / absorption [CFGK05]:
  - Clamond, D., Fructus, D., Grue, J., & Kristiansen, O. (2005). *An efficient model for three-dimensional surface wave simulations. Part II: Generation and absorption*. J. Comp. Phys., **205**(2), 686-705.
- Moving bottom case [FG07]:
  - Fructus, D., & Grue, J. (2007). *An explicit method for the nonlinear interaction between water waves and variable and moving bottom topography*. J. Comp. Phys., **222**, 720-739.

Evolution equations

In the following presentation we will follow [FCKG05]

Notations:

$\varphi(\underline{x}, t) := \phi(\underline{x}, \eta(\underline{x}, t), t)$: potential at the free surface

$V_n := \sqrt{1 + |\boldsymbol{\nabla}\eta|^2} \frac{\partial \phi}{\partial n}\Big|_{y=\eta}$: normal velocity at the free surface

Dynamic boundary conditions:

$$\eta_t = V_n$$

$$\varphi_t + g\eta + \tfrac{1}{2}\tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\varphi - \tfrac{1}{2}\tilde{v} V_n = \sigma \boldsymbol{\nabla} \cdot \left[\frac{\boldsymbol{\nabla}\eta}{\sqrt{1 + |\boldsymbol{\nabla}\eta|^2}} \right]$$

$$\tilde{\boldsymbol{u}} = \frac{\boldsymbol{\nabla}\varphi - V_n \boldsymbol{\nabla}\eta + (\boldsymbol{\nabla}\eta \times \boldsymbol{\nabla}\varphi) \times \boldsymbol{\nabla}\eta}{1 + |\boldsymbol{\nabla}\eta|^2}, \qquad \tilde{v} = \frac{V_n + \boldsymbol{\nabla}\eta \cdot \boldsymbol{\nabla}\varphi}{1 + |\boldsymbol{\nabla}\eta|^2}$$

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HOW TO FIND THE NORMAL VELOCITY V_n?

BOUNDARY INTEGRAL EQUATION METHOD (BIEM)

- The Green function is constructed by the method of images ($h = const$):

$$\int_{\mathcal{S}} \left(\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}_B} \right) \frac{\partial \phi'}{\partial n} \mathrm{d}S' = 2\pi\varphi + \int_{\mathcal{S}} \varphi' \frac{\partial}{\partial n} \left(\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}_B} \right) \mathrm{d}S'$$

- $R := |\underline{x} - \underline{x}'|$, $\tilde{r} := R^2 + (y - y')^2$, $\tilde{r}_B := R^2 + (y' + y + 2h)^2$
- $\varphi = \varphi(\underline{x}, t)$, $\varphi' = \varphi(\underline{x}', t)$
- For non-overturning surfaces: $\mathrm{d}S' = \sqrt{1 + |\nabla \eta'|^2}\, \mathrm{d}\underline{x}'$

INTRODUCE NEW VARIABLE:

- $D := \frac{\eta' - \eta}{R}$, $D \to 1/R$ as $R \to \infty$ and $D \to \frac{\partial \eta}{\partial R}$ as $R \to 0$

- Boundary Integral Equation in finite depth:

$$\int \frac{V_n'}{(1+D^2)^{1/2}}\frac{\mathrm{d}\underline{x}'}{R} + \int \frac{V_n'}{(1+4hD_BR_B^{-1}+D_B^2)^{1/2}}\frac{\mathrm{d}\underline{x}'}{R_B} = 2\pi\varphi +$$

$$\int \frac{\varphi'(R\cdot\nabla\eta' - \eta' + \eta)}{(1+D^2)^{3/2}}\frac{\mathrm{d}\underline{x}'}{R^3} + \int \frac{\varphi'(R\cdot\nabla\eta' - \eta' - \eta - 2h)}{(1+4hD_BR_B^{-1}+D_B^2)^{3/2}}\frac{\mathrm{d}\underline{x}'}{R_B^3}$$

- Using the following relation [CG01]:

$$\frac{R\cdot\nabla\eta' - \eta' + \eta}{R^3} = -\nabla'\cdot\left[(\eta'-\eta)\nabla'\frac{1}{R}\right]$$

- and switching for *simplicity* in deep water ($h\to\infty$):

$$\int V_n' R^{-1}\mathrm{d}\underline{x}' = 2\pi\varphi + \int (\eta'-\eta)\nabla'\varphi'\cdot\nabla' R^{-1}\mathrm{d}\underline{x} +$$

$$\int V_n' R^{-1}\left[1-(1+D^2)^{-1/2}\right]\mathrm{d}\underline{x}' +$$

$$\int \varphi'\left[1-(1+D^2)^{-3/2}\right]\nabla'\cdot\left[(\eta'-\eta)\nabla' R^{-1}\right]\mathrm{d}\underline{x}'$$

- Boundary Integral Equation in finite depth:

$$\int \frac{V_n'}{(1+D^2)^{1/2}} \frac{\mathrm{d}\underline{x}'}{R} + \int \frac{V_n'}{(1+4hD_B R_B^{-1} + D_B^2)^{1/2}} \frac{\mathrm{d}\underline{x}'}{R_B} = 2\pi\varphi +$$

$$\int \frac{\varphi'(R\cdot\nabla\eta' - \eta' + \eta)}{(1+D^2)^{3/2}} \frac{\mathrm{d}\underline{x}'}{R^3} + \int \frac{\varphi'(R\cdot\nabla\eta' - \eta' - \eta - 2h)}{(1+4hD_B R_B^{-1} + D_B^2)^{3/2}} \frac{\mathrm{d}\underline{x}'}{R_B^3}$$

$$\int V_n' R^{-1} \mathrm{d}\underline{x}' = \underbrace{2\pi\varphi}_{(I)} + \underbrace{\int (\eta' - \eta)\nabla'\varphi' \cdot \nabla' R^{-1} \mathrm{d}\underline{x}}_{(II)} +$$

$$\underbrace{\int V_n' R^{-1} \left[1 - (1+D^2)^{-1/2}\right] \mathrm{d}\underline{x}'}_{(III)} +$$

$$\underbrace{\int \varphi' \left[1 - (1+D^2)^{-3/2}\right] \nabla' \cdot \left[(\eta' - \eta)\nabla' R^{-1}\right] \mathrm{d}\underline{x}'}_{(IV)}$$

DECOMPOSITION OF THE INTEGRAL EQUATION: $V_n := V_n^{(1)} + V_n^{(2)} + V_n^{(3)} + V_n^{(4)}$

- $\int V_n'^{(1)} R^{-1} \mathrm{d}\underline{x}' = 2\pi\varphi$
- $\int V_n'^{(2)} R^{-1} \mathrm{d}\underline{x}' = \int (\eta' - \eta) \boldsymbol{\nabla}'\varphi' \cdot \boldsymbol{\nabla}' R^{-1} \mathrm{d}\underline{x}$
- $\int V_n'^{(3)} R^{-1} \mathrm{d}\underline{x}' =$
  $\int \varphi' \left[1 - (1 + D^2)^{-3/2} \right] \boldsymbol{\nabla}' \cdot \left[(\eta' - \eta) \boldsymbol{\nabla}' R^{-1} \right] \mathrm{d}\underline{x}'$
- $\int V_n'^{(4)} R^{-1} \mathrm{d}\underline{x}' = \int V_n' R^{-1} \left[1 - (1 + D^2)^{-1/2} \right] \mathrm{d}\underline{x}'$

- LHS can be computed by convolutions:

$$\mathcal{F}\left\{ \int V_n'^{(j)} R^{-1} \mathrm{d}\underline{x}' \right\} = 2\pi k^{-1} \int V_n'^{(j)} \mathrm{e}^{-\mathrm{i}\underline{k}\cdot\underline{x}} \mathrm{d}\underline{x}' = 2\pi k^{-1}\, \hat{V}_n^{(j)}$$

since

- $\mathcal{F}\{R^{-1}\} = 2\pi k^{-1} \mathrm{e}^{-\mathrm{i}\underline{k}\cdot\underline{x}}$

We turn to the Fourier space:

Computation of integrals

- $\hat{V}_n^{(1)} = k\hat{\varphi}$
- $\hat{V}_n^{(2)} = -k\mathcal{F}\{\eta V_n^{(1)}\} - \mathrm{i}\underline{k}\cdot\mathcal{F}\{\eta\nabla\varphi\}$
- $2\pi\hat{V}_n^{(3)} = k\mathcal{F}\left\{\int\varphi'\left[1-(1+D^2)^{-3/2}\right]\nabla'\cdot\left[(\eta'-\eta)\nabla' R^{-1}\right]\mathrm{d}\underline{x}'\right\}$
- $2\pi\hat{V}_n^{(4)} = k\mathcal{F}\left\{V_n' R^{-1}\left[1-(1+D^2)^{-1/2}\right]\right\}$

$V_n^{(4)}$ can be equivalently rewritten:

$$k\mathcal{F}\left\{\int V_n' R^{-1}\tfrac{1}{2}D^2\mathrm{d}\underline{x}'\right\} + k\mathcal{F}\left\{\int V_n' R^{-1}\left[1-\tfrac{1}{2}D^2-(1+D^2)^{-1/2}\right]\mathrm{d}\underline{x}'\right\}$$

Apply convolution formula:

$$k\mathcal{F}\left\{\int V_n' R^{-1}\frac{1}{2}D^2\mathrm{d}\underline{x}'\right\} = -\pi k\mathcal{F}\{\eta^2\mathcal{F}^{-1}\{k\hat{V}_n\}\} - 2\mathcal{F}\{\eta\mathcal{F}^{-1}\{k\mathcal{F}\{\eta V_n\}\}\} + k\mathcal{F}\{\eta^2 V_n\}$$

- $V_n^{(3)} \sim \int \varphi' \left[1 - (1 + D^2)^{-3/2}\right] \nabla' \cdot \left[(\eta' - \eta)\nabla' R^{-1}\right] \mathrm{d}\underline{x}' \sim \mathcal{O}(R^{-4})$, as $R \to \infty$
- $V_n^{(4)} \sim \int V_n' R^{-1} \left[1 - \frac{1}{2} D^2 - (1 + D^2)^{-1/2}\right] \mathrm{d}\underline{x}' \sim \mathcal{O}(R^{-5})$, as $R \to \infty$

- Apply trapezoidal rule on a few wavelengths λ

COMPUTATION OF $V_n^{(4)}$:

- Done iteratively: $V_n^{(4)} \leftarrow V_n^{(1)} + V_n^{(2)} + V_n^{(3)}$
- Convergence is achieved in max 3 iterations [FCKG05]

- Hybrid method between Taylor expansions and BIEM
- The “rests”are computed exactly (fastly decaying integrals)

NUMERICAL RESULTS - I

STOKES WAVE PROPAGATION: $ak = 0.2985$

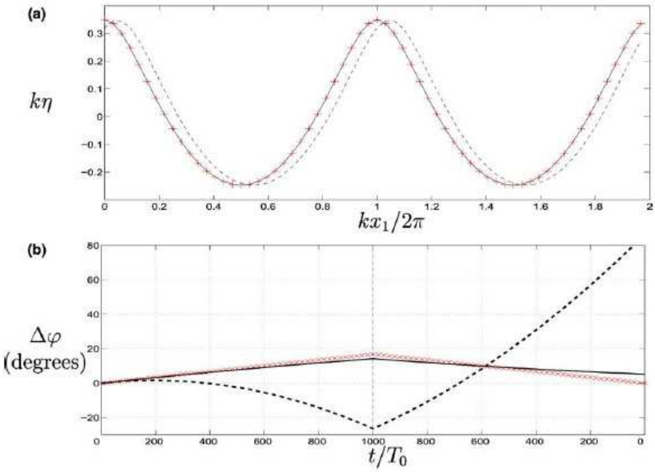

FIGURE: Iter = 3, Tol = 10^{-6} ($- - -$), Tol = 10^{-7} (—), Tol = 10^{-8} ($\times$)

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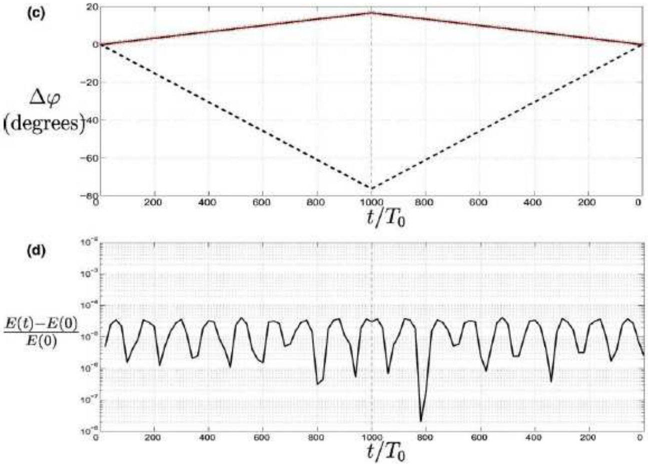

FIGURE: Tol = 10^{-8}, Iter = 1 ($---$), Iter = 3 (—), Iter = 5 ($\times$)

$$\underline{k}_a + \underline{k}_b = 3\underline{k}_0$$
$$\omega_a + \omega_b = 3\omega_0$$

Class II instability following McLean's description [McL82]

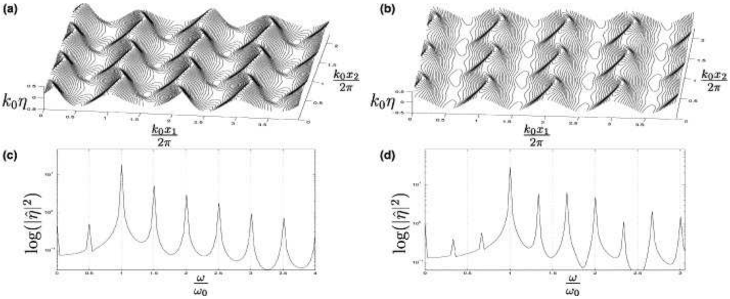

FIGURE: *Free surface elevation at two different times and corresponding Fourier spectra.*

CONCLUSIONS

PROS:

- Spectral **accuracy**
- Uses low-order Taylor expansions $\longrightarrow$ numerical **stability**
- Method was extended for generation/absorption/general bathymetries

CONS:

- As any spectral method — **periodic domains**
- Not easily accessible for beginners
- Currently no operational code is available

Thank you for your attention!

See you tomorrow at 10 a.m.!

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