MOVING LOAD ON A FLOATING ICE LAYER

DENYS DUTYKH

Director: FREDERIC DIAS

July 6, 2005

Contents

List of Figures

1. Introduction

A major motivation for the study of a moving load on a flexible beam or plate has been its application to transport systems (rail tracks, roads or runways), originally in temperate lands and subsequently in cold regions, where, in particular, floating ice sheets may be exploited.

Wave propagation in a water-ice system attracted considerable attention in the literature both as a theoretical subject as well as due to its importance to the ice engineering. From the theoretical point of view models of a water-ice system allow convenient application of mathematical tools for studying wave phenomena under realistic physical conditions. At the same time, from the engineering standpoint there are questions of great practical significance in which the wave propagation is of importance. These include stress control of the ice cover in the neighborhoods of facilities built upon ice, performance of ice breaking ships, damage of offshore constructions by floating ice sheets, spontaneous appearance of large scale cracks in the ice cover of water basins etc.

There exists a significant body of literature on linear as well as nonlinear wave propagation in a water-ice system. One of the first theoretical studies of the response of floating ice to moving loads was given in

[7]. More recently, a number of authors: Kheisin (1963, 1971) [9, 10], Nevel (1970) [18], Marchenko (1988) [12], Marchenko and Semenov (1994) [13], Milinazzo (1995, 2004) [15, 31] and others have studied the effects of a moving load on floating ice. In these investigations, the ice sheet is treated as a thin plate of infinite extent supported below by water of uniform, finite depth and the load is assumed to move with constant speed. Kheisin in [9] examines the steady motion of a point and a line load.

We cannot omit the recent monograph by Squire et al. [27] that is devoted to the rich topic of moving loads on ice plates.

Recently, a stability analysis of a water-ice system, in which ice was modeled as an elastic homogeneous layer of finite thickness, was carried out by Brevdo and Il'ichev [2]. In that work, it was shown that any 2-D homogeneous floating ice layer of infinite horizontal extension is exponentially stable.

There is also an article [19] by E. Parau and F. Dias. They modeled the ice as a Kirchhoff-Love plate and the water was assumed to be an ideal incompressible fluid. This article was restricted to the effects of nonlinearity when the load speed is close to the minimum phase speed. A weakly nonlinear analysis, based on dynamical systems theory and on normal forms, was performed.

The thin plate assumption for the ice, while producing a considerable simplification in the model, causes at the same time an exclusion of the effects related to the elastic deformations present inside the ice sheet; within this assumption only flexural deformations are accounted for.

The compressibility of water has to be taken into account for the reason of consistency. In fact the compressibility of water cannot be neglected in treating the wave dynamics in the model because the infinite medium dilatational and equivoluminal wave velocities in ice are comparable with the speed of the acoustic waves in water.

In the present work we consider the ice to be a finite thickness elastic plate. We take into account the compressibility of the water and the moving load is supposed to be a rigid block.

2. Formulation of the problem

In the present work we consider the problem of ice layer deformation under a moving block (see Figure 1). We make the following assumptions. First of all we neglect the gravitational forces. In other words there are no exterior mass forces in this model[1]. The problem is two-dimensional. The horizontal axis is denoted by x while the vertical axis is denoted by y. The x-axis is along the interface between ice and water. The ice thickness is $h = const$. We suppose that the ice layer lies on compressible water of constant depth H. The water layer

[1] The validity of this assumption is discussed below.

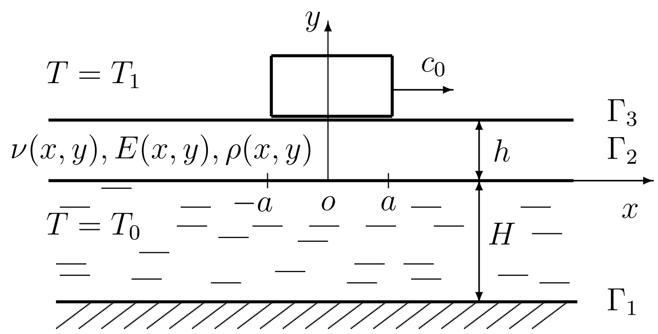

FIGURE 1. Moving load on an ice layer

lies on a rigid half-space. In general the mechanical properties (density, Poisson ratio, Young's modulus) of the ice layer are functions of $(x, y) \in \mathbb{R} \times [0, h]$. In the next section we will discuss this point. A rigid block moves along the x-axis with constant velocity c_0. In this work we take into account the friction between the block and the ice. The block load is modeled by a pressure patch applied to the ice upper boundary:

$$L(x, t) = \begin{cases} l(x), & |x - c_0 t| < a \\ 0, & \text{else} \end{cases}$$

A similar expression can be written for the friction between the ice layer and the block:

$$G(x, t) = \begin{cases} g(x), & |x - c_0 t| < a \\ 0, & \text{else} \end{cases}$$

Also we suppose that the water temperature and the air temperature are T_0 and $T_1 = const$ respectively. It is important to take into account the heat transfer because the mechanical properties of ice depend on its temperature.

3. Ice mechanical properties

In this section we will briefly review only a few aspects of this very complicated subject. The best general reference here is [27].

Experiments show that Young's modulus depends on the fractional brine volume[2] ν_b:

$$\boxed{E = 10 - 3.5\nu_b}$$

In this empirical formula, E is expressed in GPa.

[2]To our knowledge "fractional brine volume" means the volume fraction of brine in the ice.

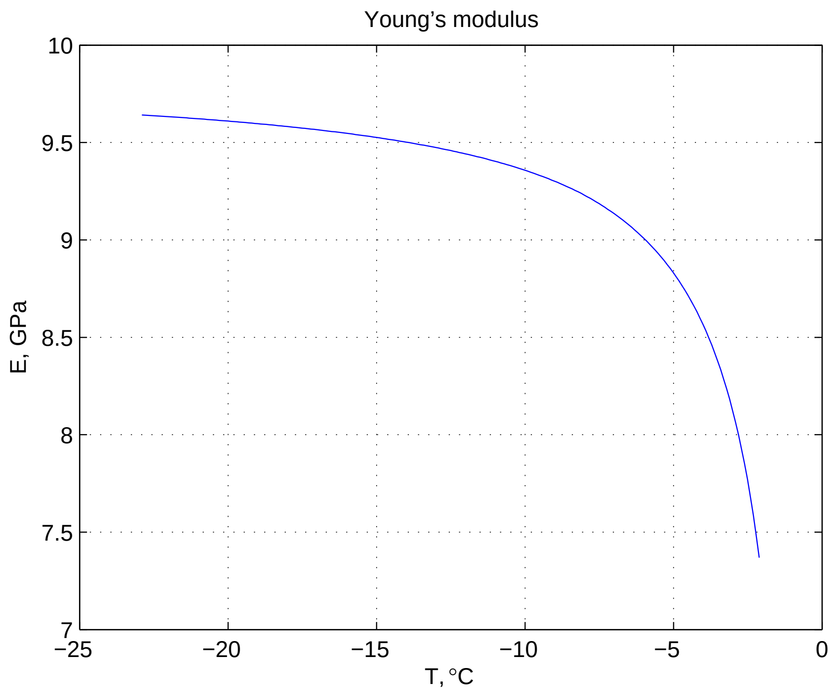

FIGURE 2. Young's modulus dependence on the ice temperature

Next, the fractional brine volume depends on the water salinity and temperature through the following formula:

$$\nu_b = \begin{cases} \frac{S}{1000}\left(\frac{52.56}{|T|} - 2.28\right), & -2.06°C < T < -0.5°C \\ \frac{S}{1000}\left(\frac{45.917}{|T|} + 0.93\right), & -8.2°C < T < -2.06°C \\ \frac{S}{1000}\left(\frac{43.795}{|T|} + 1.189\right), & -22.9°C < T < -8.2°C \end{cases}$$

where S is the ice salinity in $^\circ/_{\circ\circ}$.

The Young's modulus dependence on the ice temperature is shown in Figure 2.

As for Poisson's ratio, experiments show that this parameter does not vary much in the ice with a typical value

$$\boxed{\nu = 0.33 \pm 0.03}$$

Anyhow we will assume this parameter to be a function of x and y since it does not change our solution technique.

The density of bubble-free ice is also a function of temperature. Experiments show that the density ρ is almost a linear function of temperature. Using this information and the fact that at $T = 0°C$ $\rho = 916.5kg/m^3$ and at $T = -30°C$ $\rho = 920.7kg/m^3$ we can reconstruct this functional dependence:

$$\boxed{\rho(T) = -0.14T + 916.5}$$

where T in $^\circ C$, ρ in kg/m^3.

4. Lamé's equations for an inhomogeneous medium

In this section we review the well-known Lamé's equations when the mechanical properties of the material are functions of space.

The best general reference here is [29].

First of all we make some remarks on the notation. Let us denote by

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix}$$

the stress tensor components and by

$$\begin{pmatrix} \varepsilon_x & \gamma_{xy} \\ \gamma_{yx} & \varepsilon_y \end{pmatrix}$$

the deformation tensor components.

The dynamic equations are

$$\left\{ \begin{array}{rl} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho X = & \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho Y = & \rho \frac{\partial^2 v}{\partial t^2}, \end{array} \right. \tag{1}$$

where (u, v) is the displacement field and (X, Y) is the volume force field.

The kinematic equations are

$$\left\{ \begin{array}{rlrl} \varepsilon_x = & \frac{\partial u}{\partial x}, & \gamma_{xy} = & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \\ \varepsilon_y = & \frac{\partial v}{\partial y}, & \gamma_{yx} = & \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{array} \right. \tag{2}$$

and the physical equations are

$$\left\{ \begin{array}{rl} \sigma_x = & 2\mu\varepsilon_x + \lambda\theta, \\ \tau_{xy} = & \mu\gamma_{xy}, \\ \sigma_y = & 2\mu\varepsilon_y + \lambda\theta \end{array} \right. \tag{3}$$

where $\theta = \varepsilon_x + \varepsilon_y$. Using (3) we obtain the following derivatives of the stress tensor components:

$$\begin{array}{rl} \frac{\partial \sigma_x}{\partial x} = & 2\mu \frac{\partial^2 u}{\partial x^2} + \lambda \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + 2 \frac{\partial \mu}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial \lambda}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ \frac{\partial \tau_{xy}}{\partial y} = & \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \frac{\partial \mu}{\partial y} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \frac{\partial \tau_{yx}}{\partial x} = & \mu \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \right) + \frac{\partial \mu}{\partial x} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \frac{\partial \sigma_y}{\partial y} = & 2\mu \frac{\partial^2 v}{\partial y^2} + \lambda \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) + 2 \frac{\partial \mu}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial \lambda}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \end{array} \tag{4}$$

Now we substitute equalities (4) into (1) and obtain Lamé's equations:

$$(5)\quad \begin{aligned} (\lambda+2\mu)\tfrac{\partial^2 u}{\partial x^2}+\mu\tfrac{\partial^2 u}{\partial y^2}+(\lambda+\mu)\tfrac{\partial^2 v}{\partial x\partial y}+ &\\ +\left(\tfrac{\partial\lambda}{\partial x}+2\tfrac{\partial\mu}{\partial x}\right)\tfrac{\partial u}{\partial x}+\tfrac{\partial\lambda}{\partial x}\tfrac{\partial v}{\partial y}+\tfrac{\partial\mu}{\partial y}\left(\tfrac{\partial u}{\partial y}+\tfrac{\partial v}{\partial x}\right)+\rho X = &\quad \rho\tfrac{\partial^2 u}{\partial t^2}\\ \mu\tfrac{\partial^2 v}{\partial x^2}+(\lambda+2\mu)\tfrac{\partial^2 v}{\partial y^2}+(\lambda+\mu)\tfrac{\partial^2 u}{\partial x\partial y}+\left(\tfrac{\partial\lambda}{\partial y}+2\tfrac{\partial\mu}{\partial y}\right)\tfrac{\partial v}{\partial y}+ &\\ +\tfrac{\partial\lambda}{\partial y}\tfrac{\partial u}{\partial x}+\tfrac{\partial\mu}{\partial x}\left(\tfrac{\partial u}{\partial y}+\tfrac{\partial v}{\partial x}\right)+\rho Y = &\quad \rho\tfrac{\partial^2 v}{\partial t^2} \end{aligned}$$

If Lamé's constants depend only on y equations (5) become simpler:

$$\begin{aligned} (\lambda+2\mu)\tfrac{\partial^2 u}{\partial x^2}+\mu\tfrac{\partial^2 u}{\partial y^2}+(\lambda+\mu)\tfrac{\partial^2 v}{\partial x\partial y}+\tfrac{\partial\mu}{\partial y}\left(\tfrac{\partial u}{\partial y}+\tfrac{\partial v}{\partial x}\right)+\rho X = &\quad \rho\tfrac{\partial^2 u}{\partial t^2}\\ \mu\tfrac{\partial^2 v}{\partial x^2}+(\lambda+2\mu)\tfrac{\partial^2 v}{\partial y^2}+(\lambda+\mu)\tfrac{\partial^2 u}{\partial x\partial y}+\left(\tfrac{\partial\lambda}{\partial y}+2\tfrac{\partial\mu}{\partial y}\right)\tfrac{\partial v}{\partial y}+ &\\ +\tfrac{\partial\lambda}{\partial y}\tfrac{\partial u}{\partial x}+\rho Y = &\quad \rho\tfrac{\partial^2 v}{\partial t^2} \end{aligned}$$

Finally one can rewrite equations (5) in vector form:

$$\begin{aligned} &(\lambda+\mu)\underline{\operatorname{grad}}\operatorname{div}\underline{u}+\mu\Delta\underline{u}+\operatorname{div}\underline{u}\cdot\underline{\operatorname{grad}}\lambda+\\ &\qquad +(\underline{\underline{\operatorname{grad}}}\,\underline{u}+{}^t\,\underline{\underline{\operatorname{grad}}}\,\underline{u})\cdot\underline{\operatorname{grad}}\mu+\rho\underline{F}=\rho\frac{\partial^2\underline{u}}{\partial t^2} \end{aligned}$$

If the mechanical properties are constant then one can recognize the classical Lamé's equations:

$$(6)\quad \boxed{(\lambda+\mu)\underline{\operatorname{grad}}\operatorname{div}\underline{u}+\mu\Delta\underline{u}+\rho\underline{F}=\rho\frac{\partial^2\underline{u}}{\partial t^2}}$$

5. International Equation of State of Seawater

In this work we need to know the sound velocity in water. By definition the velocity γ of propagation of acoustic waves is given by

$$\gamma^2=\frac{\partial p}{\partial\rho}=\frac{1}{\beta\rho}$$

where β is the compressibility factor of sea water and p the pressure.

The speed of sound can be obtained from the International Equation of State of Seawater (IES80):

$$\rho(S,T,p)=\frac{\rho(S,T,0)}{1-p/K(S,T,p)}$$

with

$$\begin{aligned} \rho(S,T,0)=\ &999.842594+6.793952\times10^{-2}T-9.09529\times10^{-3}T^2+\\ &1.001685\times10^{-4}T^3-1.120083\times10^{-6}T^4+6.536332\times10^{-9}T^5+\\ &8.24493\times10^{-1}S-4.0899\times10^{-3}TS+7.6438\times10^{-5}T^2S\\ &-8.2467\times10^{-7}T^3S+5.3875\times10^{-9}T^4S-5.72466\times10{-}3S^{\frac{3}{2}}\\ &+1.0227\times10^{-4}TS^{\frac{3}{2}}-1.6546\times10^{-6}T^2S^{\frac{3}{2}}+4.8314\times10^{-4}S^2 \end{aligned}$$

and

$$K(S,T,p) = 19652.21 + 148.4206T - 2.327105T^2 + 1.360447\times 10^{-2}T^3 - 5.155288\times 10^{-3}T^4 + 3.239908p + 1.43713\times 10^{-3}Tp + 1.16092\times 10^{-4}T^2p - 5.77905\times 10^{-7}T^3p + 8.50935\times 10^{-5}p^2 - 6.12293\times 10^{-6}Tp^2 + 5.2787\times 10^{-8}T^2p^2 + 54.6746S - 0.603459TS + 1.09987\times 10^{-2}T^2S - 6.167\times 10^{-5}T^3S + 7.944\times 10^{-2}S^{\frac{3}{2}} + 1.6483\times 10^{-2}TS^{\frac{3}{2}} - 5.3009\times 10^{-4}T^2S^{\frac{3}{2}} + 2.2838\times 10^{-3}pS - 1.0981\times 10^{-5}TpS - 1.6078\times 10^{-6}T^2pS + 1.91075\times 10^{-4}pS^{\frac{3}{2}} - 9.9348\times 10^{-7}p^2S + 2.0816\times 10^{-8}Tp^2S + 9.1697\times 10^{-10}T^2p^2S$$

But there are several simplified formulae. In the present work we will make use of the following one:

$$\gamma(S,T,p) = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 1.58\times 10^{-6}p \tag{7}$$

where γ is the speed of sound in m/s, T is temperature in $^\circ C$, S is salinity and p is pressure in Pa.

In general we consider the pressure to be hydrostatic[3] and that is why we can replace the last term in (7) by $-0.016z$ where z is the water depth in m. Thus we have

$$\gamma(z) = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) - 0.016z$$

See [24] for more details.

In order to obtain a PDE with constant coefficients for water we replace $\gamma(z)$ by its average value:

$$\gamma = \frac{1}{H}\int_{-H}^{0}\gamma(z)dz = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.008H$$

where H is water depth.

6. Linear acoustics in perfect fluids

The author was inspired by [2, 3] to use this model in the present work.

[3]We can make this assumption since we deal with linear acoustics approximation in the water. Let us recall that this model implies pressure perturbation $p(x,t)$ to be such that $p(x,t) = \varepsilon(x,t)p_0$ with $|\varepsilon(x,t)| \ll 1$ and p_0 - hydrostatic pressure. For further discussion the reader should see section 6 and more general reference [25].

In this section we closely follow [25]. We try to model sea water by considering it as an ideal fluid.

6.1. **Basic hypotheses.** We consider the unsteady flow of a fluid under the following assumptions:

(1) the fluid is perfect (inviscid $\underline{\underline{\tau}} = \underline{\underline{0}}$),
(2) the fluid is not a heat conductor ($\underline{q} = \underline{0}$),
(3) there are no exterior mass forces,
(4) the flow is continuous.

The flow is denoted by

$$(\underline{V} + \underline{v}, p_0 + p, \rho_0 + \rho). \tag{8}$$

It is written as a perturbation of the steady flow $(\underline{V}, p_0, \rho_0)$ where $\underline{V}$ is the velocity vector, p_0 is the pressure and ρ_0 is the water density. It is assumed that the perturbation $(\underline{v}, p, \rho)$ is such that

$$\begin{aligned} \underline{v}(x,t) &= \varepsilon_1(x,t)\underline{V}, &\quad \text{where} \quad & 0 \le |\varepsilon_1(x,t)| \ll 1, \\ p(x,t) &= \varepsilon_2(x,t)p_0, &\quad \text{where} \quad & 0 \le |\varepsilon_2(x,t)| \ll 1, \\ \rho(x,t) &= \varepsilon_3(x,t)\rho_0, &\quad \text{where} \quad & 0 \le |\varepsilon_3(x,t)| \ll 1. \end{aligned}$$

6.2. **Linear acoustics approximation.** Our perturbed flow (8) satisfies Euler's equations:

Conservation of mass:

$$\frac{\partial}{\partial t}(\rho_0 + \rho) + \operatorname{div}[(\rho_0 + \rho)(\underline{V} + \underline{v})] = 0, \tag{9}$$

Momentum conservation:

$$(\rho_0 + \rho)\left\{\frac{\partial}{\partial t}(\underline{V} + \underline{v}) + \underline{\operatorname{grad}}\,(\underline{V} + \underline{v})(\underline{V} + \underline{v})\right\} = -\underline{\operatorname{grad}}\,(p_0 + p), \tag{10}$$

Energy conservation:

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{(\rho_0 + \rho)\left[e' + \frac{1}{2}(\underline{V} + \underline{v})^2\right]\right\} + \\ &+ \operatorname{div}\left[(\rho_0 + \rho)(e' + \frac{1}{2}(\underline{V} + \underline{v})^2)(\underline{V} + \underline{v})\right] = -\operatorname{div}[(p_0 + p)(\underline{V} + \underline{v})] \end{aligned}$$

where e' is specific internal energy.

The fluid pressure is a function of density $\rho_0 + \rho$ and entropy[4] s_0. Linearizing the state equation around the basic state (ρ_0, s_0) yields

$$p(x,t) = \gamma^2 \rho(x,t), \qquad \gamma^2 := \left(\frac{\partial p}{\partial \rho}\right)_s (\rho_0, s_0). \tag{11}$$

[4]In this work we restrict ourselves to homoentropic flows

The basic flow $(p_0, \rho_0, \underline{V})$ being steady and uniform, the linearisation of equations (9), (10) is

$$\left(\frac{\partial}{\partial t} + \underline{V} \cdot \underline{\text{grad}}\right) \rho + \rho_0 \,\text{div}\, \underline{v} = 0, \tag{12}$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{\underline{\text{grad}}}\, \underline{v} \cdot \underline{V} = -\frac{1}{\rho_0} \underline{\text{grad}}\, p. \tag{13}$$

From equation (12) one finds

$$\text{div}\, \underline{v} = -\frac{1}{\rho_0} \left(\frac{\partial}{\partial t} + \underline{V} \cdot \underline{\text{grad}}\right) \rho. \tag{14}$$

Taking the divergence of equation (13) and substituting into (14) yields

$$\frac{1}{\gamma^2} \left\{\frac{\partial}{\partial t} + \underline{V} \cdot \underline{\text{grad}}\right\} \left(\frac{\partial \rho}{\partial t} + \underline{V} \cdot \underline{\text{grad}}\, \rho\right) = \Delta \rho$$

If we multiply the last equality by γ^4 and use (11) we obtain

$$\boxed{\left\{\frac{\partial}{\partial t} + \underline{V} \cdot \underline{\text{grad}}\right\} \left(\frac{\partial p}{\partial t} + \underline{V} \cdot \underline{\text{grad}}\, p\right) = \gamma^2 \Delta p}$$

If we assume that before perturbation the fluid was motionless ($\underline{V} = \underline{0}$) we obtain the equation that we will work with:

$$\boxed{\frac{1}{\gamma^2} \frac{\partial^2 p}{\partial t^2} = \Delta p} \tag{15}$$

To this equation we should add the boundary condition that the accelerations of solid and fluid coincide at the interface:

$$\rho_g \frac{\partial^2 v}{\partial t^2} = \frac{\partial p}{\partial y} \tag{16}$$

where ρ_g is the ice density at the bottom and v is the vertical displacement.

7. Complete model

In this section we put together all the equations obtained above and discuss initial and boundary conditions.

First of all we start by the homogeneous problem: all mechanical properties of the ice are the same everywhere. We recall here different relations among mechanical constants. These relations were taken from [26].

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{17}$$

$$\mu = \frac{E}{2(1+\nu)}, \tag{18}$$

$$E = \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu},$$
$$\nu = \frac{\lambda}{2(\lambda + \mu)}.$$

The displacement field in the ice induced by the moving load satisfies Lamé-Clapeyron equations in $\mathbb{R} \times (0, h)$:

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} = \rho_g\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial T}{\partial x}, \tag{19}$$

$$\mu\frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} = \rho_g\frac{\partial^2 v}{\partial t^2} + \rho_g g + \alpha\frac{\partial T}{\partial y} \tag{20}$$

where α is the linear expansion coefficient. The temperature $T(x, y)$ in the ice layer satisfies the heat conduction equation in the same region $\mathbb{R} \times (0, h)$:

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} + \eta\frac{\partial}{\partial t}\operatorname{div}\underline{u} = \Delta T + \frac{1}{\kappa}Q. \tag{21}$$

where κ is the thermal conductivity coefficient, $\eta = \gamma T_0/k$ with T_0 the temperature of undeformed body, $\gamma := \frac{E}{1-2\nu}\alpha$ and k is the heat conductivity coefficient. Equation (21) is needed if we consider thermoelastic effects. Later we will make further simplifications and take into account thermoeffects only in the ice mechanical properties. There is a technical reason: the terms due to thermal expansion and gravitation do not belong to $L_2(\mathbb{R})$ and prevent the author to use integral transform methods or more exactly Fourier integral transforms. Let us see now, how to avoid this difficulty.

In water the pressure below the ice is given by the linear acoustic equation (15):

$$\frac{1}{\gamma^2}\frac{\partial^2 p}{\partial t^2} = \Delta p$$

Next we describe initial and boundary conditions. Let us start with initial conditions. Since we look for travelling-wave type solutions this point can be omitted.

To describe the boundary conditions we use the notation

$$\Gamma_1 = \{(x, y) \in \mathbb{R}^2 | y = -H\},$$
$$\Gamma_2 = \{(x, y) \in \mathbb{R}^2 | y = 0\},$$
$$\Gamma_3 = \{(x, y) \in \mathbb{R}^2 | y = h\}.$$

Along Γ_3 we have the following boundary conditions:

$$\sigma_y(x, h) = l(x), \tag{22}$$
$$\tau_{xy}(x, h) = g(x), \tag{23}$$
$$T(x, h) = T_1. \tag{24}$$

The physical meaning of the first equation is to impose the load on the upper ice boundary. The second one corresponds to given friction between the ice layer and the moving load. The third condition imposes air temperature along the ice-air interface. In fact we suppose this temperature to be constant along Γ_3.

On Γ_2 we have:

$$\sigma_y(x,0) = -p(x,0) + \rho_w g v(x,0), \tag{25}$$

$$\tau_{xy}(x,0) = 0, \tag{26}$$

$$\rho_g \frac{\partial^2 v}{\partial t^2} = \frac{\partial p}{\partial y}, \tag{27}$$

$$T(x,0) = T_0, \tag{28}$$

where g is the gravity acceleration and $v(x,0)$ is the vertical displacement. The first condition is the pressure balance at the ice-water interface. The second condition means that we neglect the friction between ice and water. The third one was already discussed (16). The last one is clear.

On Γ_1 we just have one condition:

$$\frac{\partial p}{\partial y} = 0 \tag{29}$$

In fact this is the same condition as (16) in the case where the boundary is fixed.

Now we simplify our model. First we start by equations (19),(20). In these equations we neglect thermal expansion terms and gravitational forces. To justify this simplification we have to look at dimensionless equations. For example we consider equation (20) which is more interesting because it contains both terms. In non-dimensional form this equation is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\lambda + 2\mu}{\mu}\frac{\partial^2 v}{\partial y^2} + \frac{\lambda+\mu}{\mu}\frac{\partial^2 u}{\partial x \partial y} = \frac{\rho_g H^2}{\mu \tau^2}\frac{\partial^2 v}{\partial t^2} + \frac{\rho_g g H}{\mu} + \frac{\alpha T_0}{\mu H}\frac{\partial T}{\partial y}$$

where H, τ, T_0 are characteristic length (water depth), time and temperature respectively.

The gravitational term is of order $10^{-6} H/m$. Thus, for physically relevant values[5] of H this coefficient is small. Consequently we can neglect the effect of gravity. In [2] the authors explain this fact by the smallness of the characteristic velocity $\sqrt{g(H+h)}$ of free surface gravity waves in a water layer of depth $H < 5000m$ compared to the infinite medium dilatational wave velocity in ice $\sqrt{\frac{\lambda+2\mu}{\rho_g}}$, or, in other words the water wave velocity is much smaller than the body wave velocity. Finally, the thermal expansion term is of order 10^{-13} because for ice $\alpha \approx 44(\pm 9) \times 10^{-6} K^{-1}$. This value was taken from [28].

[5] $H \leq 1000m$

Then we simplify the heat-conduction equation (21). First in ice there are no internal heat sources, thus $Q = 0$. The next term to be neglected is the thermoelastic one. In fact we decouple the thermoelastic problem and temperature is only needed to determine the ice mechanical properties at each point. Then we find the steady solution of this equation. This can be justified by the fact that boundary conditions are time independent and that this process has taken infinite time before our consideration. The last assumption concerning this equation is that the temperature T depends only on the variable y: $T = T(y)$. It is natural to make this assumption because boundary conditions are invariant by translations along the x-axis.

With all these assumptions one can write the solution of the simplified heat-conduction equation (21):

$$\boxed{T(y) = T_0\left(1 - \frac{y}{h}\right) + T_1\frac{y}{h}} \tag{30}$$

Now we come to the problem that we are going to solve in the next section:

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} = \rho_g\frac{\partial^2 u}{\partial t^2}, \tag{31}$$

$$\mu\frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} = \rho_g\frac{\partial^2 v}{\partial t^2}, \tag{32}$$

$$\frac{1}{\gamma^2}\frac{\partial^2 p}{\partial t^2} = \Delta p. \tag{33}$$

To these equations we have to add the boundary conditions discussed earlier. We repeat one more time that the absence of initial conditions is explained by the fact that we look for travelling wave solutions.

The general plan followed by the author is to:

(1) Solve problem (31), (32), (33) for an homogeneous ice layer
(2) Generalize this solution to a multi-layer case as a special case of inhomogeneity
(3) Take the limit of an infinite number of layers (consequently, at this stage, we will obtain the solution of an inhomogeneous problem)
(4) If time permits, solve the quasi-dynamic contact problem

8. Fourier transform

In this work we often use integral Fourier transforms. For a good practical reference on this subject we invite the reader to look at [21]. If the reader is interested in theoretical aspects of Fourier transforms we suggest the reading of [4]. We use the following definition:

Definition 8.1. *If $f(x)$ is an absolutely integrable function on $\mathbb{R}$ (i.e., $\int_{-\infty}^{+\infty} |f(x)|dx < \infty$), then the direct Fourier transform of $f(x)$, $\mathfrak{F}[f]$,* *and the inverse Fourier transform of $f(x)$, $\mathfrak{F}^{-1}[f]$ are the functions given by*

$$\mathfrak{F}[f](s) = \int_{-\infty}^{+\infty} f(x)e^{-ixs}dx \tag{34}$$

and

$$\mathfrak{F}^{-1}[f](s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{ixs}dx \tag{35}$$

Definition 8.2. *A function $f : \mathbb{R} \to \mathbb{R}(\mathbb{C})$ is said to be "classically transformable" if either*

(1) *f is absolutely integrable on $\mathbb{R}$, or*
(2) *f is the Fourier transform (or inverse Fourier transform) of an absolutely integrable function, or*
(3) *f is a linear combination of an absolutely integrable function and a Fourier transform (or inverse Fourier transform) of an absolutely integrable function.*

If f is a classically transformable but not absolutely integrable function, then it can be shown that formulas (34) and (35) can still be used to define $\mathfrak{F}[f]$ and $\mathfrak{F}^{-1}[f]$ provided that the integrals are taken in the sense of Cauchy principal values.

We suppose that solutions of equations from Section 7 are classically transformable. It is a necessary condition to use integral transform methods.

8.1. **Discrete Fourier transform.** We are interested in discrete Fourier transforms in order to calculate numerically the inverse integral Fourier transform. This topic is discussed in [22] but the author does not agree with their way to calculate integral (35). Here we should add some additional remarks. In [22], on page 503, one can find the formula:

$$H(f_n) = \int_{-\infty}^{\infty} h(t)e^{2\pi i f_n t}dt \approx \sum_{k=0}^{N-1} h_k e^{2\pi i f_n t_k}\Delta = \Delta \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}$$

where $f_n := \frac{n}{N\Delta}$ and $t_k := k\Delta$. We cannot understand why the interval $(-\infty, 0)$ is not taken into consideration. That is why in the present work we show how one should perform this discretization. This formula is used in the programs given in Appendix 13.1.

Before starting the Fourier integral discretization we recall the formulae of discrete Fourier transform used in very well known scientific computing libraries [14], [32]. Incidentally this formula is used in MatLab to perform FFT computations.

Direct transform:

$$X_k = \sum_{j=1}^{N} x_j \omega_N^{(j-1)(k-1)}.$$

Inverse transform:

(36) $$x_j = \frac{1}{N} \sum_{k=1}^{N} X_k \omega_N^{-(j-1)(k-1)}.$$

where $\omega_N = e^{\frac{-2\pi i}{N}}$.

Now we calculate numerically this integral:

(37) $$u(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}(\omega) e^{i\omega x} d\omega$$

To perform our discretization we need several notation: δ denotes the discretization step in Fourier space domain and k_j is a wave number. Therefore

$$\omega_j := j\delta = 2\pi k_j,$$

$$x_n := \frac{n}{N\Delta}$$

where $\Delta := \frac{\delta}{2\pi}$.

$$\begin{aligned}
u(x_n) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}(\omega) e^{i\omega x_n} d\omega = \\
= \int_{0}^{+\infty} \hat{u}(2\pi k) e^{2\pi i k x_n} dk - \int_{0}^{+\infty} \hat{u}(-2\pi k) e^{-2\pi i k x_n} dk \approx \\
\approx \int_{0}^{N\delta} \hat{u}(2\pi k) e^{2\pi i k x_n} dk - \int_{0}^{N\delta} \hat{u}(-2\pi k) e^{-2\pi i k x_n} dk \approx \\
\approx \Delta \left(\sum_{j=0}^{N-1} \hat{u}(2\pi k_j) e^{2\pi i k_j x_n} - \sum_{j=0}^{N-1} \hat{u}(-2\pi k_j) e^{-2\pi i k_j x_n} \right) = \\
= \Delta \left(\sum_{j=0}^{N-1} \hat{u}_j e^{2\pi i j n/N} - \sum_{j=0}^{N-1} \hat{u}_{-j} e^{-2\pi i j n/N} \right) = \\
= \Delta \left(\sum_{j=1}^{N} \hat{u}_{j-1} \omega_N^{-(j-1)n} - \sum_{j=1}^{N} \hat{u}_{-(j-1)} \omega_N^{(j-1)n} \right).
\end{aligned}$$

If we recall the definitions of direct and inverse discret Fourier transforms we can calculate the sums in the last equality

$$\boxed{\{u(x_n)\}_{n=-N/2+1}^{N/2} = \Delta\big(N \,\mathrm{ifft}\,(\{\hat{u}_{j+N}\}_{j=0}^{N-1}) - \mathrm{fft}\,(\{\hat{u}_{N-j}\}_{j=0}^{N-1})\big)}$$

8.2. **Filon's quadrature formulae.** There is an alternative way to compute Fourier type integrals with a special class of quadrature formulae. These formulae are named Filon's formulae because Filon was first to propose this idea. Unfortunately we do not know the original paper. We discovered this method from [1].

These formulae are particulary useful for highly oscillating functions. The main idea of the method is not to interpolate the integrated function but to interpolate its amplitude with a polynomial of low order. In this work we show how to do it in the case of linear interpolation. The reader can obtain better results by using quadratic interpolation. One more advantage of this method is that one can better approximate integrals by changing the order of interpolation but with the same number of discretization points. With FFT based methods, one can obtain more accurate results only by adding discretization points. For some problems the difference can be important. But the computational complexity of Filon's quadrature is $O(N^2)$ operations while the method described in the previous subsection performs it with $O(N \ln N)$ operations.

We calculate numerically the integral

$$I = \int_a^b f(x)e^{i\omega x}\, dx \equiv \int_a^b F(x)\, dx.$$

We use the subintervals

$$a = x_0 < x_1 < \ldots < x_N = b.$$

On the k^{th} subinterval we replace $f(x)$ by its interpolation polynomial with $q+1$ nodes:

$$I_N = \sum_{k=1}^{N} I_k = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} P_q(x)e^{i\omega x}\, dx.$$

Filon obtained his formula with $q = 2$ (quadratic interpolation). In the present work we deal with linear interpolation:

$$P_1(x) = f(x_{k-1}) + (x - x_{k-1})f(x_{k-1}, x_k) = y_{k-1} + \frac{y_k - y_{k-1}}{x_k - x_{k-1}}(x - x_{k-1})$$

where $f(x_{k-1}, x_k)$ denotes the first divided difference of the function f at the points x_{k-1}, x_k[6].

[6]A notion of divided differences is very common in Soviet numerical analysis literature. The definition of the first divided difference is $f(x_{k-1}, x_k) \equiv \frac{y_k - y_{k-1}}{x_k - x_{k-1}}$. For second-order divided differences we have a recurrent relation: $f(x_{k-1}, x_k, x_{k+1}) \equiv$

Then we have:

$$I_k = \int_{x_{k-1}}^{x_k} P_1(x)e^{i\omega x}\,dx =$$

$$= \int_{x_{k-1}}^{x_k} \left(y_{k-1} + \frac{y_k - y_{k-1}}{x_k - x_{k-1}}(x - x_{k-1}) \right) e^{i\omega x}\,dx =$$

$$= y_{k-1}\frac{e^{i\omega x}}{i\omega}\Big|_{x_{k-1}}^{x_k} + \frac{y_k - y_{k-1}}{x_k - x_{k-1}} \left\{ (x - x_{k-1})\frac{e^{i\omega x}}{i\omega}\Big|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} \frac{e^{i\omega x}}{i\omega}\,dx \right\} =$$

$$= \frac{y_{k-1}}{i\omega}(e^{i\omega x_k} - e^{i\omega x_{k-1}}) + (y_k - y_{k-1})\frac{e^{i\omega x_k}}{i\omega} - \frac{y_k - y_{k-1}}{x_k - x_{k-1}}\frac{e^{i\omega x}}{(i\omega)^2}\Big|_{x_{k-1}}^{x_k} =$$

$$= -\frac{F_{k-1}}{i\omega} + \frac{F_k}{i\omega} + \frac{y_k - y_{k-1}}{\omega^2 h_k}(e^{i\omega x_k} - e^{i\omega x_{k-1}}) =$$

$$= \frac{F_k - F_{k-1}}{i\omega} + 2i\sin\left(\frac{\omega h_k}{2}\right)\left(\frac{y_k - y_{k-1}}{\omega^2 h_k}\right) e^{i\omega x_{k-1/2}},$$

where $F_k \equiv y_k e^{i\omega x_k}$, $h_k \equiv x_k - x_{k-1}$.

Summation over k gives

$$\boxed{I_N = \sum_{k=1}^{N} I_k = \frac{F_N - F_0}{i\omega} + \frac{2i}{\omega^2}\sum_{k=1}^{N}\frac{\sin\omega h_k/2}{h_k}(y_k - y_{k-1})e^{i\omega x_{k-1/2}}}$$

To conclude this subsection the author would like to mention here that the above formula is useful only in the case where $\omega(b-a) \gg 1$. To explain this we can look at the last formula and see that it requires a lot of operations per discretization step (more than for Simpson's formula). In the case where $\omega(b-a) \ll 1$ the function $F(x)$ is no longer highly oscillating and we suggest to use another quadrature formulae.

9. Homogeneous ice layer

9.1. **Water pressure determination.**

First we begin with linear acoustic equation. We repeat the problem that we are going to solve:

$$\frac{1}{\gamma^2}\frac{\partial^2 p}{\partial t^2} = \Delta p, \qquad (x, y) \in \mathbb{R} \times (-H, 0),$$

$$\rho_g \frac{\partial^2 v}{\partial t^2} = \frac{\partial p}{\partial y}, \qquad y = 0,$$

$$\frac{\partial p}{\partial y} = 0, \qquad y = -H.$$

$\frac{f(x_{k+1},x_k) - f(x_k,x_{k-1})}{x_{k+1} - x_{k-1}}$. And the definition of divided differences of k^{th} order is: $f(x_i, x_{i+1}, \ldots, x_{i+k}) = \frac{f(x_{i+1},\ldots,x_k) - f(x_i,\ldots,x_{k-1})}{x_{i+k} - x_i}$. These differences are useful to approximate derivatives.

In this work we look for travelling wave solutions of our equations. In mathematical notation this means

$$p(x, y, t) := p(x - c_0 t, y)$$

where c_0 is the moving load velocity. We substitute this particular choice of the solution in (15) and obtain:

$$\chi \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{38}$$

where χ denotes the dimensionless quantity:

$$\chi := 1 - \frac{c_0^2}{\gamma^2}$$

It is interesting to note that equation (38) is

- elliptic if $|c_0| < |\gamma|$
- hyperbolic, otherwise.

In this work we consider only the first case since it is the most physically relevant. Recall that the speed of sound in water is $\gamma \approx 1500 m/s$.

Let us multiply equation (38) by $e^{-i\omega x}$ and integrate on $\mathbb{R}$:

$$\frac{d^2 \hat{p}}{dy^2} - \chi \omega^2 \hat{p} = 0$$

We have assumed that we are in the elliptic case. In other words $0 < \chi \leq 1$. A general solution of this equation can be written as

$$\hat{p}(\omega, y) = C_1(\omega) \cosh(\sqrt{\chi}\omega y) + C_2(\omega) \sinh(\sqrt{\chi}\omega y)$$

We take the Fourier transform of the boundary condition (29) on Γ_1:

$$\frac{\partial p}{\partial y} = 0 \Rightarrow \frac{d\hat{p}}{dy} = 0, \qquad y = -H$$

Next we use this condition in order to establish a relation between the unknown functions $C_1(\omega)$ and $C_2(\omega)$:

$$\frac{d\hat{p}}{dy} = C_1(\omega)\sqrt{\chi}\omega \sinh(\sqrt{\chi}\omega y) + C_2(\omega)\sqrt{\chi}\omega \cosh(\sqrt{\chi}\omega y)$$

Evaluating it at $y = -H$ yields

$$C_2(\omega) = \tanh(\sqrt{\chi}\omega H) C_1(\omega).$$

Now there is only one independent function:

$$\hat{p}(\omega, y) = C_1(\omega)(\cosh(\sqrt{\chi}\omega y) + \tanh(\sqrt{\chi}\omega H) \sinh(\sqrt{\chi}\omega y))$$

The second condition (33) on Γ_2 is

$$y = 0 : \frac{\partial p}{\partial y} = \rho_g \frac{\partial^2 v}{\partial t^2}.$$

We take the Fourier transform of this equation and substitute the travelling wave solution $v(x,y,t) := v(x - c_0 t, y)$:

$$\frac{d\hat{p}}{dy} = -\rho_g c_0^2 \omega^2 \hat{v}(\omega, 0)$$

Finally we can find the unknown function $C_1(\omega)$:

$$\left.\frac{d\hat{p}}{dy}\right|_{y=0} = C_1(\omega)\sqrt{\chi}\omega \tanh(\sqrt{\chi}\omega H) = -\rho_g c_0^2 \omega^2 \hat{v}(\omega, 0)$$

$$C_1(\omega) = -\frac{\rho_g \omega c_0^2}{\sqrt{\chi}} \coth(\sqrt{\chi}\omega H)\hat{v}(\omega, 0)$$

The solution for $\hat{p}$ is

$$\hat{p}(\omega, y) = -\frac{\rho_g \omega c_0^2}{\sqrt{\chi}} \coth(\sqrt{\chi}\omega H)\hat{v}(\omega, 0)(\cosh(\sqrt{\chi}\omega y) + \\ + \tanh(\sqrt{\chi}\omega H)\sinh(\sqrt{\chi}\omega y)) \tag{39}$$

9.2. **Solution of Lamé's equations for an homogeneous layer.** Recall that Lamé's equations are given by (31) (32). In order to obtain dimensionless equations below we need to divide equation (31) by $\lambda + 2\mu$ and (32) by μ:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\mu}{\lambda + 2\mu}\frac{\partial^2 u}{\partial y^2} + \frac{\lambda + \mu}{\lambda + 2\mu}\frac{\partial^2 v}{\partial x \partial y} = \frac{\rho_g}{\lambda + 2\mu}\frac{\partial^2 u}{\partial t^2}, \tag{40}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\lambda + 2\mu}{\mu}\frac{\partial^2 v}{\partial y^2} + \frac{\lambda + \mu}{\mu}\frac{\partial^2 u}{\partial x \partial y} = \frac{\rho_g}{\mu}\frac{\partial^2 v}{\partial t^2}. \tag{41}$$

Let us introduce the velocities:

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho_g}} = \sqrt{\frac{E(1-\nu)}{\rho_g(1+\nu)(1-2\nu)}} \approx 3900 m/s$$

and

$$c_2 = \sqrt{\frac{\mu}{\rho_g}} = \sqrt{\frac{E}{2\rho_g(1+\nu)}} \approx 1900 m/s.$$

These constants represent the propagation velocities of elastic waves in two orthogonal directions x and y.

Again, we look for travelling-wave solutions:

$$u(x,y,t) := u(x - c_0 t, y),$$

$$v(x,y,t) := v(x - c_0 t, y)$$

where c_0 is the moving load velocity.

After substituting this particular form in equations (40) and (41), and using the relations between different elastic constants (17) and (18), one obtains:

$$\left(1-\frac{c_0^2}{c_1^2}\right)\frac{\partial^2 u}{\partial x^2}+\frac{1-2\nu}{2(1-\nu)}\frac{\partial^2 u}{\partial y^2}+\frac{1}{2(1-\nu)}\frac{\partial^2 v}{\partial x\partial y}=0$$
$$\left(1-\frac{c_0^2}{c_2^2}\right)\frac{\partial^2 v}{\partial x^2}+\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 v}{\partial y^2}+\frac{1}{1-2\nu}\frac{\partial^2 u}{\partial x\partial y}=0$$

We introduce the notation

$$\chi_i := 1-\frac{c_0^2}{c_i^2}, \qquad i=1,2.$$

In Fourier space, one has a system of ODE

$$\frac{d^2\hat{u}}{dy^2}-\frac{i\omega}{1-2\nu}\frac{d\hat{v}}{dy}-\frac{2(1-\nu)}{1-2\nu}\chi_1\omega^2\hat{u}=0$$
$$\frac{d^2\hat{v}}{dy^2}-\frac{i\omega}{2(1-\nu)}\frac{d\hat{u}}{dy}-\frac{1-2\nu}{2(1-\nu)}\chi_2\omega^2\hat{u}=0$$

One can rewrite this system as a first order system by introducing the new functions $\hat{w}:=\frac{d\hat{u}}{dy}$ and $\hat{z}:=\frac{d\hat{v}}{dy}$:

$$\begin{cases} \frac{d\hat{u}}{dy} &= \hat{w} \\ \frac{d\hat{w}}{dy} &= \frac{2(1-\nu)}{1-2\nu}\chi_1\omega^2\hat{u}+\frac{i\omega}{1-2\nu}\hat{z} \\ \frac{d\hat{v}}{dy} &= \hat{z} \\ \frac{d\hat{z}}{dy} &= \frac{i\omega}{2(1-\nu)}\hat{w}+\frac{1-2\nu}{2(1-\nu)}\chi_2\omega^2\hat{v} \end{cases}$$

or in matrix form

$$\frac{d\hat{X}}{dy}=A\hat{X}$$

where

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2(1-\nu)}{1-2\nu}\chi_1\omega^2 & 0 & 0 & \frac{i\omega}{1-2\nu} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{i\omega}{2(1-\nu)} & \frac{1-2\nu}{2(1-\nu)}\chi_2\omega^2 & 0 \end{pmatrix},$$
$$\hat{X}=(\hat{u},\hat{w},\hat{v},\hat{z})^t$$

The general solution of this system of ODE depends on four functions:

(42) $$\hat{u}(\omega,y)=C_1(\omega)\cosh(k_1\omega y)+C_2(\omega)\sinh(k_1\omega y)+ \\ +C_3(\omega)\sinh(k_2\omega y)+C_4(\omega)\cosh(k_2\omega y),$$

(43) $$\hat{v}(\omega,y)=\frac{i}{k_1^2}\big(k_1\sinh(k_1\omega y)C_1(\omega)+k_1\cosh(k_1\omega y)C_2(\omega)+ \\ +F\cosh(k_2\omega y)C_3(\omega)+F\sinh(k_2\omega y)C_4(\omega)\big).$$

The notation

$$k_1 := \sqrt{1 - \frac{2c_0^2\rho_g(1+\nu)}{E}}$$

$$k_2 := \sqrt{1 - \frac{c_0^2\rho_g}{E(1-\nu)}(1-\nu-2\nu^2)}$$

$$F := k_2 k_1^2$$

has been introduced.

We would like to make a remark. From equality (43) it is clear that the ice deflection tends to infinity when $k_1 \to 0$. The block velocity corresponding to $k_1 = 0$ is

$$c_0 = \sqrt{\frac{E}{2\rho_g(1+\nu)}} = c_2$$

Thus we found analytically a resonant velocity.

Recall that all boundary conditions are written in terms of the stress tensor.That is why we need to determine the two components of this tensor σ_y, τ_{xy} in order to build below a system of linear equations with unknown functions $C_1(\omega), C_2(\omega), C_3(\omega), C_4(\omega)$. The connection between the displacements and the stress tensor is given by Hooke's law

$$\sigma_y = \lambda\frac{\partial u}{\partial x} + (\lambda + 2\mu)\frac{\partial v}{\partial y},$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

We take Fourier transforms of these two equations

$$\hat{\sigma}_y = -i\lambda\omega\hat{u} + (\lambda+2\mu)\frac{d\hat{v}}{dy} \tag{44}$$

$$\hat{\tau}_{xy} = \mu\left(\frac{d\hat{u}}{dy} - i\omega\hat{v}\right) \tag{45}$$

and substitute the solutions (42), (43) in (44), (45):

$$\hat{\sigma}_y = iE\omega\big(B_1\cosh(k_1\omega y)C_1(\omega) + B_1\sinh(k_1\omega y)C_2(\omega) + \\ + B_2\sinh(k_2\omega y)C_3(\omega) + B_2\cosh(k_2\omega y)C_4(\omega)\big)$$

where

$$B_1 := \frac{1}{1+\nu}, \qquad B_2 := \frac{(1-\nu)Fk_2 - \nu k_1^2}{k_1^2(1+\nu)(1-2\nu)} = \frac{(1-\nu)k_2^2 - \nu}{(1+\nu)(1-2\nu)}.$$

$$\hat{\tau}_{xy} = \mu\omega(A_1\sinh(k_1\omega y)C_1(\omega) + A_1\cosh(k_1\omega y)C_2(\omega) + \\ + A_2\cosh(k_2\omega y)C_3(\omega) + A_2\sinh(k_2\omega y)C_4(\omega))$$

where

$$A_1 \equiv A := k_1 + \frac{1}{k_1}, \qquad A_2 := k_2 + \frac{F}{k_1^2} = 2k_2.$$

We rewrite the expressions for $\hat{u}$, $\hat{v}$, $\hat{\sigma}_y$, $\hat{\tau}_{xy}$ in vector form:

$$\begin{pmatrix} \hat{u}(\omega, y) \\ \hat{v}(\omega, y) \\ \hat{\sigma}_y(\omega, y) \\ \hat{\tau}_{xy}(\omega, y) \end{pmatrix} = I \cdot M(\omega, y) \begin{pmatrix} C_1(\omega) \\ C_2(\omega) \\ C_3(\omega) \\ C_4(\omega) \end{pmatrix} \tag{46}$$

where

$$I = \operatorname{diag}\{I_{11}, I_{22}, I_{33}, I_{44}\} = \operatorname{diag}\left\{1, i/k_1^2, iE\omega, \mu\omega\right\}$$

and $M(\omega, y)$ is a $4x4$ real matrix with components:

$$\begin{aligned}
m_{11}(\omega, y) &= \cosh(k_1\omega y), & m_{12}(\omega, y) &= \sinh(k_1\omega y), \\
m_{13}(\omega, y) &= \sinh(k_2\omega y), & m_{14}(\omega, y) &= \cosh(k_2\omega y), \\
m_{21}(\omega, y) &= k_1 \sinh(k_1\omega y), & m_{22}(\omega, y) &= k_1 \cosh(k_1\omega y), \\
m_{23}(\omega, y) &= F \cosh(k_2\omega y), & m_{24}(\omega, y) &= F \sinh(k_2\omega y), \\
m_{31}(\omega, y) &= B_1 \cosh(k_1\omega y), & m_{32}(\omega, y) &= B_1 \sinh(k_1\omega y), \\
m_{33}(\omega, y) &= B_2 \sinh(k_2\omega y), & m_{34}(\omega, y) &= B_2 \cosh(k_2\omega y), \\
m_{41}(\omega, y) &= A_1 \sinh(k_1\omega y), & m_{42}(\omega, y) &= A_1 \cosh(k_1\omega y), \\
m_{43}(\omega, y) &= A_2 \cosh(k_2\omega y), & m_{44}(\omega, y) &= A_2 \sinh(k_2\omega y).
\end{aligned}$$

In order to solve this problem we have to find the four unknown functions $C_1(\omega), C_2(\omega), C_3(\omega), C_4(\omega)$. We will obtain these unknowns by using the boundary conditions.

On Γ_3:

$$\sigma_y(x, h) = l(x),$$

or, in Fourier space,

$$\hat{\sigma}_y(\omega, h) = \hat{l}(\omega).$$

From this we obtain the first equation:

$$\hat{\sigma}_y(\omega, h) = I_{33} \sum_{j=1}^{4} m_{3j}(\omega, h) C_j(\omega) = \hat{l}(\omega),$$

The same equation in expanded form reads

$$\begin{aligned}
&B_1 \cosh(k_1\omega h) C_1(\omega) + B_1 \sinh(k_1\omega h) C_2(\omega) + \\
&\qquad + B_2 \sinh(k_2\omega h) C_3(\omega) + B_2 \cosh(k_2\omega h) C_4(\omega) = -i\frac{\hat{l}(\omega)}{E\omega}
\end{aligned}$$

In this work we take for numerical computations a constant load concentrated on the segment $[-a, a]$:

$$l(x) = \begin{cases} P, & |x| < a \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier transform of this function is

$$\hat{l}(\omega) = P\frac{\sin(\omega a)}{\omega}.$$

Then the first equation takes the form:

$$B_1 \cosh(k_1\omega h)C_1(\omega) + B_1 \sinh(k_1\omega h)C_2(\omega)+ \\ + B_2 \sinh(k_2\omega h)C_3(\omega) + B_2 \cosh(k_2\omega h)C_4(\omega) = -i\frac{P\sin(\omega a)}{E\omega^2} \tag{47}$$

Similarly the second condition on Γ_3 is

$$\tau_{xy}(x, h) = g(x)$$

or, in Fourier space,

$$\hat{\tau}_{xy}(\omega, h) = \hat{g}(\omega)$$

In the present work we take $g(x) = 0$ for the numerical computations (in other words we neglect the friction between the moving load and the ice) in order to be able to compare our results with those of other researchers. Recall that the more general case $g(x) \neq 0$ can be treated here as well.

In the general case the short hand version of the second equation is

$$\hat{\tau}_{xy}(\omega, h) = I_{44}\sum_{j=1}^{4} m_{4j}(\omega, h)C_j(\omega) = \hat{g}(\omega),$$

or, in expanded form,

$$A_1 \sinh(k_1\omega h)C_1(\omega) + A_1 \cosh(k_1\omega h)C_2(\omega)+ \\ + A_2 \cosh(k_2\omega h)C_3(\omega) + A_2 \sinh(k_2\omega h)C_4(\omega) = \hat{g}(\omega). \tag{48}$$

The next equation is obtained from the condition on Γ_2 that there is no friction between water and ice. In fact the generalization is not difficult. One has

$$\tau_{xy}(x, 0) = 0 \Rightarrow \hat{\tau}_{xy}(\omega, 0) = 0,$$

which yields

$$I_{44}\sum_{j=1}^{4} m_{4j}(\omega, 0)C_j(\omega) = 0,$$

or, in expanded form,

$$A_1C_2(\omega) + A_2C_3(\omega) = 0. \tag{49}$$

The last equation is slightly more difficult to obtain. On Γ_2 we have the condition

$$\sigma_y(x, 0) = -p(x, 0) + \rho_w g v(x, 0),$$

or, in Fourier space,

$$\hat{\sigma}_y(\omega, 0) = -\hat{p}(\omega, 0) + \rho_g g\hat{v}(\omega, 0).$$

Above we obtained the formula (39):

$$\hat{p}(\omega, y) = -\frac{\omega\rho_g c_0^2}{\sqrt{\chi}} \coth(\sqrt{\chi}\omega H)\hat{v}(\omega, 0)(\cosh(\sqrt{\chi}\omega y)+ \\ + \tanh(\sqrt{\chi}\omega H)\sinh(\sqrt{\chi}\omega y))$$

For $y = 0$, it becomes

$$\hat{p}(\omega, 0) = -\frac{\omega\rho_g c_0^2}{\sqrt{\chi}} \coth(\sqrt{\chi}\omega H)\hat{v}(\omega, 0)$$

The last equation follows:

$$\hat{\sigma}_y(\omega, 0) = \left(\frac{\rho_g\omega c_0^2}{\sqrt{\chi}} \coth(\sqrt{\chi}\omega H) + \rho_w g\right)\hat{v}(\omega, 0)$$

We introduce the short-hand notation

$$s(\omega) = \left(\frac{\rho_g\omega c^2}{\sqrt{\chi}} \coth(\sqrt{\chi}\omega H) + \rho_w g\right).$$

Therefore, the last equation is

$$\sum_{j=1}^{4}(m_{3j}(\omega, 0) - I_{22}/I_{33}s(\omega)m_{2j}(\omega, 0))C_j(\omega) = 0.$$

or, if we write the expressions for coefficients m_{ij},

$$B_1C_1(\omega) - \frac{s(\omega)}{E\omega k_1}C_2(\omega) - \frac{Fs(\omega)}{E\omega k_1^2}C_3(\omega) + B_2C_4(\omega) = 0. \tag{50}$$

We have four unknown functions and four equations (47), (48), (49), (50). Thus, solving this linear system allows us to reconstruct the solution in Fourier space. The last step is to take the inverse Fourier transform. We do it with an FFT algorithm.

The author would like to mention here that the matrix of this system is ill-conditioned for ω in the neighborhood of zero and for large ω. This problem generates several purely numerical difficulties. In our program we distinguish two cases $\omega \ll 1$ and $\omega \gg 1$. In the first case we multiply the last equation by ω in order to avoid the division on ω and by $\sinh(\sqrt{\chi}\omega H)$ because $s(\omega)$ is singular at zero[7]. For large ω we divide the first and the second equations by $\cosh(k_1\omega h)$ in order to avoid the operations with very large numbers. As an illustration, we give here the limit of the linear system matrix when $\omega \to 0$:

$$\lim_{\omega\to 0} A(\omega) = \begin{pmatrix} B_1 & 0 & 0 & B_2 \\ 0 & A_1 & A_2 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

[7] In fact $s(\omega)$ contains $\coth(\sqrt{\chi}\omega H)$ which tends to ∞ when $\omega \to 0$

Appendix 13.2 shows the results of the numerical computations for different values of ice thickness and moving load velocity (Figures 11 - 18). The program is given in Appendix 13.1.

The values of physical parameters can be found in this table:

Moving block length	$2a = 2m$
Ice thickness	$h = 1.5m$
Water depth	$H = 5.0m$
Poisson ratio	$\nu = 0.33$
Young's modulus	$E = 9.5Gpa$
The standard gravitational acceleration	$g = 9.80665m/s^2$
Block velocity	$c_0 = 15.0m/s$
Sound velocity in the water	$\gamma = 1500m/s$
Ice density	$\rho_g = 926kg/m^3$
Water density	$\rho_w = 1027kg/m^3$
Block load	$F = 17500Pa$

9.3. **Gravitation influence on an homogeneous layer deformation.** In this section we are going to investigate the influence of the gravitational force on an homogeneous layer deformation.

Let us consider the governing equations. In the ice layer we have Lamé's equations with a gravitational term

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} = \rho_g \frac{\partial^2 u}{\partial t^2} \tag{51}$$

$$\mu\frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} = \rho_g \frac{\partial^2 v}{\partial t^2} + \rho_g g \tag{52}$$

In the water domain fortunately we have the same equation

$$\frac{1}{\gamma^2}\frac{\partial^2 p}{\partial t^2} = \Delta p.$$

In fact, this is due to the fact that we write the equation (15) for the perturbations of the steady flow and this steady flow contains already the gravitational term.

The boundary conditions are the same too.

Recall that we could not apply directly the integral transforms method to the equations (51) (52) because they contain the term $\rho_g g$ which is not $L_2(-\infty, +\infty)$ integrable.

In order to avoid this technical difficulty we apply the method of solutions superposition. In other words, we use a very strong property of over problem - the linearity.

We represent the solution of the equations (51) (52) in the following form

$$u(x, y) = u_1(x, y) + u_2(x, y),$$
$$v(x, y) = v_1(x, y) + v_2(x, y).$$

The displacements field (u_1, v_1) satisfies the homogeneous equations but inhomogeneous boundary conditions

$$(\lambda + 2\mu)\frac{\partial^2 u_1}{\partial x^2} + \mu\frac{\partial^2 u_1}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v_1}{\partial x \partial y} = \rho_g \frac{\partial^2 u_1}{\partial t^2},$$

$$\mu\frac{\partial^2 v_1}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v_1}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u_1}{\partial x \partial y} = \rho_g \frac{\partial^2 v_1}{\partial t^2}.$$

The boundary conditions are

$$\sigma_y^1(x, h) = l(x),$$

$$\tau_{xy}^1(x, h) = f(x),$$

$$\sigma_y^1(x, 0) = -p(x, 0) + \rho_w g v_1(x, 0),$$

$$\tau_{xy}^1(x, 0) = 0.$$

It is clear that we already solved this problem in the previous section.

The displacements field (u_2, v_2) satisfies inhomogeneous equations and homogeneous boundary conditions

$$(\lambda + 2\mu)\frac{\partial^2 u_2}{\partial x^2} + \mu\frac{\partial^2 u_2}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v_2}{\partial x \partial y} = \rho_g \frac{\partial^2 u_2}{\partial t^2},$$

$$\mu\frac{\partial^2 v_2}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v_2}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u_2}{\partial x \partial y} = \rho_g \frac{\partial^2 v_2}{\partial t^2} + \rho_g g.$$

The boundary conditions are

$$\sigma_y^2(x, h) = 0, \tag{53}$$

$$\tau_{xy}^2(x, h) = 0, \tag{54}$$

$$\sigma_y^2(x, 0) = 0, \tag{55}$$

$$\tau_{xy}^2(x, 0) = 0. \tag{56}$$

In this section we are going to determine (u_2, v_2) from these equations in order to estimate the gravity influence on the layer deformation under the moving load.

It is clear now that $(u_1 + u_2, v_1 + v_2)$ satisfies initial problem (51) (52).

Now we would like to make several physical assumptions that will simplify considerably our problem.

First of all $u_2 \equiv 0$ since the gravitation compresses only our elastic layer. Then, it is not difficult to see that $v_2(x, y, t) \equiv v_2(y, t)$. We can explain it by the symmetry of our problem. The fact that the gravitational field is constant yields that we have a steady solution $v_2 = v_2(y)$.

If we take into account all assumptions made above, we will have only an ordinary differential equation

$$(\lambda + 2\mu)\frac{d^2 v_2}{dy^2} = \rho_g g,$$

or, if we recall that $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho_g}}$, we obtain

$$\frac{d^2 v_2}{dy^2} = \frac{g}{c_1^2}.$$

Its solution is

$$v_2(y) = \frac{g}{2c_1^2}y^2 + Ay + B.$$

So, we have four boundary conditions (53) (54) (55) (56) and two unknown constants A and B. Fortunately there is no contradiction here because the boundary conditions (54) (56) are satisfied identically

$$\tau_{xy}^2(x, y) = \mu\Big(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\Big) \equiv 0.$$

We use the condition (53) to determine A:

$$\sigma_y^2(x, y) = \lambda\frac{\partial u_2}{\partial x} + (\lambda + 2\mu)\frac{\partial v_2}{\partial y} = (\lambda + 2\mu)\Big(\frac{g}{c_1^2}y + A\Big).$$

For $y = h$ it becomes

$$\sigma_y^2(x, h) = (\lambda + 2\mu)\Big(\frac{g}{c_1^2}h + A\Big)$$

which yields

$$A = -\frac{g}{c_1^2}h.$$

Unfortunately we cannot satisfy in the same time the condition (55) and we have to determine one more unknown constant B. That is why we replace the condition (55) by another

$$v_2(0) = 0. \tag{57}$$

The physical meaning of this condition is that the ice-water interface is motionless. From (57) one can easily find

$$B = 0.$$

Thus, we have the solution

$$\begin{aligned} v_2(y) &= \frac{g}{2c_1^2}y(y - 2h), \qquad y \in [0, h], \\ u_2(y) &= 0. \end{aligned}$$

We would like to analyze the obtained result. In order to estimate the gravitation influence on the layer deformation we need to estimate the function $v_2(y)$

$$|v_2(y)| \leq |v_2(h)| = \frac{g}{2c_1^2}h^2 \approx 3.125 \times 10^{-7}.$$

This estimation is true because the function $v_2(y)$ is obviously monotonous.

We conclude that the gravity effects are important since the solution (u_2, v_2) is of the same order with the solution (u_1, v_1) obtained in the previous section.

10. Multilayer generalization and Lamzyuk-Privarnikov's functions method

In this section we consider almost the same problem but the ice layer is replaced by a multilayer pack. The layers in this pack differ by their thickness and mechanical properties (density, Young's modulus, Poisson ratio). But they are constant in each layer. Within each layer we attach local coordinate system $x'O'y'$.

This problem can be considered as an approximation of real ice layers with properties depending on the depth. Figure 3 provides an illustration of this situation.

Such multilayer approximations are also used to model roads asphalt cover. In practice the engineers usually take three layers. Very seldom five.

Below we will use the results of this section to obtain the solution for an inhomogeneous layer problem.

Let $\overrightarrow{w}(\omega, y) =^t (\hat{u}(\omega, y), \hat{v}(\omega, y), \hat{\sigma_y}(\omega, y), \hat{\tau_{xy}}(\omega, y))$. In the previous section we obtained the expression (46):

$$\overrightarrow{w}(\omega, y) = IM(\omega, y)\overrightarrow{C}(\omega)$$

where the matrices M and I were defined earlier.

We introduce a new vector $\overrightarrow{\alpha}(\omega)$ which has a physical meaning:

$$\overrightarrow{w}(\omega, 0) = I\overrightarrow{\alpha}(\omega).$$

The matrix $M(\omega, 0)$ is not singular. It is can be easily seen from its expression:

$$M(\omega, 0) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & k_1 & k_2 k_1^2 & 0 \\ B_1 & 0 & 0 & B_2 \\ 0 & A_1 & 2k_2 & 0 \end{pmatrix}$$

After simplification, one obtains

$$\det M(\omega, 0) = \frac{k_1 k_2 (1 - k_1^2)(1 - k_2^2)}{1 - 2\nu} \neq 0.$$

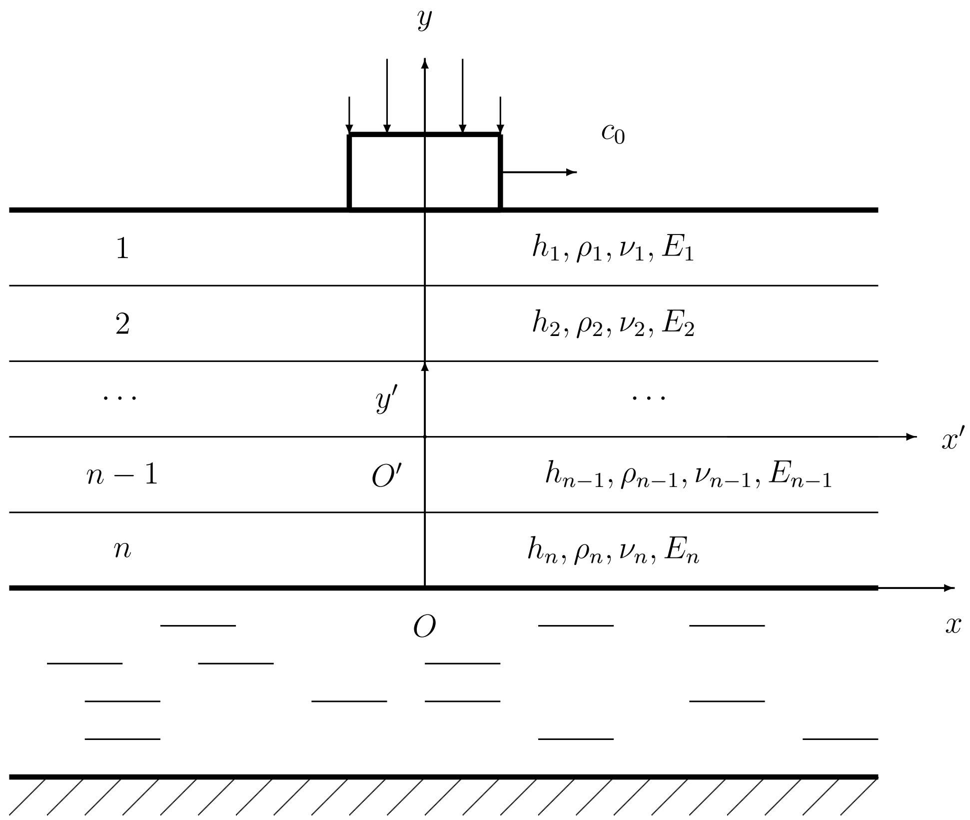

FIGURE 3. Moving load on the multilayer plate

One can easily find the connection between the vector $\overrightarrow{\alpha}$ and $\overrightarrow{C}$:

$$\overrightarrow{C}(\omega) = M^{-1}(\omega, 0)\overrightarrow{\alpha}(\omega).$$

This formula indicates that we can reduce our problem to that of finding the vector $\overrightarrow{\alpha}$, which is more convenient and natural.

Suppose we have found $\overrightarrow{\alpha}$. Then we can find the vector $\overrightarrow{w}(\omega, y)$ using this relation:

$$\overrightarrow{w}(\omega, y) = IM(\omega, y)\overrightarrow{C}(\omega) = IH(\omega, y)\overrightarrow{\alpha}(\omega)$$

with $H(\omega, y) := M(\omega, y)M^{-1}(\omega, 0) = (h_{ij}(\omega, y))_{1 \le i,j \le 4}$. It is easy to find explicitly the elements of this matrix[8]:

$$h_{11}(\omega, y) = \frac{B_1 \cosh(k_2\omega y) - B_2 \cosh(k_1\omega y)}{B_1 - B_2},$$

$$h_{12}(\omega, y) = \frac{2k_2 \sinh(k_1\omega y) - A \sinh(k_2\omega y)}{k_1 k_2 (2 - Ak_1)},$$

[8]Below A means A_1

$$h_{13}(\omega, y) = \frac{\cosh(k_1\omega y) - \cosh(k_2\omega y)}{B_1 - B_2},$$
$$h_{14}(\omega, y) = \frac{\sinh(k_2\omega y) - k_1k_2\sinh(k_1\omega y)}{k_2(2 - Ak_1)},$$
$$h_{21}(\omega, y) = \frac{B_1k_1^2k_2\sinh(k_2\omega y) - B_2k_1\sinh(k_1\omega y)}{B_1 - B_2},$$
$$h_{22}(\omega, y) = \frac{2\cosh(k_1\omega y) - Ak_1\cosh(k_2\omega y)}{2 - Ak_1},$$
$$h_{23}(\omega, y) = k_1\frac{\sinh(k_1\omega y) - k_1k_2\sinh(k_2\omega y)}{B_1 - B_2},$$
$$h_{24}(\omega, y) = k_1^2\frac{\cosh(k_2\omega y) - \cosh(k_1\omega y)}{2 - Ak_1},$$
$$h_{31}(\omega, y) = B_1B_2\frac{\cosh(k_2\omega y) - \cosh(k_1\omega y)}{B_1 - B_2},$$
$$h_{32}(\omega, y) = \frac{2B_1k_2\sinh(k_1\omega y) - AB_2\sinh(k_2\omega y)}{k_1k_2(2 - Ak_1)},$$
$$h_{33}(\omega, y) = \frac{B_1\cosh(k_1\omega y) - B_2\cosh(k_2\omega y)}{B_1 - B_2},$$
$$h_{34}(\omega, y) = \frac{B_2\sinh(k_2\omega y) - B_1k_1k_2\sinh(k_1\omega y)}{k_2(2 - Ak_1)},$$
$$h_{41}(\omega, y) = \frac{2k_2B_1\sinh(k_2\omega y) - AB_2\sinh(k_1\omega y)}{B_1 - B_2},$$
$$h_{42}(\omega, y) = 2A\frac{\cosh(k_1\omega y) - \cosh(k_2\omega y)}{k_1(2 - Ak_1)},$$
$$h_{43}(\omega, y) = \frac{A\sinh(k_1\omega y) - 2k_2\sinh(k_2\omega y)}{B_1 - B_2},$$
$$h_{44}(\omega, y) = \frac{2\cosh(k_2\omega y) - Ak_1\cosh(k_1\omega y)}{2 - Ak_1}.$$

As we established earlier the solution of our problem is completely determined by the four functions $\overrightarrow{\alpha}(\omega) =^t (\alpha(\omega), \beta(\omega), \gamma(\omega), \delta(\omega))$. In the multilayer pack with n layers we associate to each layer number j its vector $\overrightarrow{\alpha}_j(\omega)$. So the solution is determined by n vectors $\{\overrightarrow{\alpha}_j(\omega)\}_{j=1}^n$.

We assume that during deformation the layers do not detach. Thus, at the interface between two layers, we have the natural conditions of continuity:

$$\begin{aligned}
u^{(j)}(x, -h_j) &= u^{(j+1)}(x, 0),\\
v^{(j)}(x, -h_j) &= v^{(j+1)}(x, 0),\\
\sigma_y^{(j)}(x, -h_j) &= \sigma_y^{(j+1)}(x, 0)\\
\tau_{xy}^{(j)}(x, -h_j) &= \tau_{xy}^{(j+1)}(x, 0)
\end{aligned}$$

Here h_j denotes the thickness j^{th} layer. These conditions are written in a local coordinate system associated with each layer. Now we take Fourier transforms of these equalities and write them in vector form:

$$\overrightarrow{w}_j(-h_j) = \overrightarrow{w}_{j+1}(0).$$

We prefer to rewrite this identity in terms of $\overrightarrow{\alpha}$:

$$I_{j+1}H_{j+1}(\omega, 0)\overrightarrow{\alpha}_{j+1}(\omega) = I_j H_j(\omega, -h_j)\overrightarrow{\alpha}_j(\omega).$$

Note that $H(\omega, 0) = M(\omega, 0)M^{-1}(\omega, 0) = Id$. Therefore

$$\boxed{\overrightarrow{\alpha}_{j+1}(\omega) = I_{j+1}^{-1} I_j H_j(\omega, -h_j)\overrightarrow{\alpha}_j(\omega)} \tag{58}$$

If we knew the vector $\overrightarrow{\alpha}_1(\omega)$ we could find all the others recurrence (58). The problem is that the boundary conditions give us only two components of the vector $\overrightarrow{\alpha}_1$ and two components of $\overrightarrow{\alpha}_n$. Thus, we have some kind of discrete boundary problem.

With the recurrence relation (58) we can find the connection between $\overrightarrow{\alpha}_1(\omega)$ and $\overrightarrow{\alpha}_n(\omega)$:

$$\begin{aligned}\overrightarrow{\alpha}_n(\omega) &= I_n^{-1} I_{n-1} H_{n-1}(\omega, -h_{n-1})\overrightarrow{\alpha}_{n-1}(\omega) = \\ &= I_n^{-1} I_{n-1} H_{n-1}(\omega, -h_{n-1})\left(I_{n-1}^{-1} I_{n-2} H_{n-2}(\omega, -h_{n-2})\overrightarrow{\alpha}_{n-2}(\omega)\right) = \dots \\ &= \prod_{k=1}^{n-1} I_{n-k+1}^{-1} I_{n-k} H_{n-k}(\omega, -h_{n-k})\overrightarrow{\alpha}_1(\omega).\end{aligned}$$

Let

$$D = (d_{ij})_{1\le i,j\le 4} := \prod_{k=1}^{n-1} I_{n-k+1}^{-1} I_{n-k} H_{n-k}(\omega, -h_{n-k}) \in Mat_{4\times 4}(\mathbb{C}).$$

We have

$$\overrightarrow{\alpha}_n(\omega) = D(\omega, h_1, h_2, \dots, h_{n-1})\overrightarrow{\alpha}_1(\omega).$$

In Fourier space at the interface Γ_2 we have the boundary conditions

$$\begin{aligned}\hat{\tau}_{xy}^{(n)} &= 0, \\ \hat{\sigma}_y^{(n)} &= -\hat{p}(\omega, 0) + \rho_g g \hat{v}(\omega, 0) = s(\omega)\hat{v}(\omega, 0) = s(\omega)\hat{v}^{(n)}.\end{aligned}$$

By definition $\overrightarrow{w}(\omega, 0) = \overrightarrow{w}_n(\omega) = I_n\overrightarrow{\alpha}_n(\omega)$. So we can obtain the first equation:

$$\begin{aligned}\hat{\tau}_{xy}^{(n)} = \mu_n\omega\delta_n(\omega) = \mu_n\omega\big(&d_{41}(\omega, h_1, h_2, \dots, h_{n-1})\alpha_1(\omega) + \\ &+ d_{42}(\omega, h_1, h_2, \dots, h_{n-1})\beta_1(\omega) + d_{43}(\omega, h_1, h_2, \dots, h_{n-1})\gamma_1(\omega) + \\ &+ d_{44}(\omega, h_1, h_2, \dots, h_{n-1})\delta_1(\omega)\big) = 0,\end{aligned}$$

or in equivalent form:

$$\begin{aligned}&d_{41}(\omega, h_1, h_2, \dots, h_{n-1})\alpha_1(\omega) + d_{42}(\omega, h_1, h_2, \dots, h_{n-1})\beta_1(\omega) + \\ &+ d_{43}(\omega, h_1, h_2, \dots, h_{n-1})\gamma_1(\omega) + d_{44}(\omega, h_1, h_2, \dots, h_{n-1})\delta_1(\omega) = 0.\end{aligned} \tag{59}$$

Similarly we have the condition $\hat{\sigma}_y^{(n)} = s(\omega)\hat{v}^{(n)}$ that gives us one more equation:

$$
\begin{gathered}
iE_n\omega\big(d_{31}(\omega,h_1,h_2,\dots,h_{n-1})\alpha_1(\omega)+d_{32}(\omega,h_1,h_2,\dots,h_{n-1})\beta_1(\omega)+\\
+d_{33}(\omega,h_1,h_2,\dots,h_{n-1})\gamma_1(\omega)+d_{34}(\omega,h_1,h_2,\dots,h_{n-1})\delta_1(\omega)\big)=\\
=s(\omega)\frac{i}{k_{1n}^2}\big(d_{21}(\omega,h_1,h_2,\dots,h_{n-1})\alpha_1(\omega)+\\
+d_{22}(\omega,h_1,h_2,\dots,h_{n-1})\beta_1(\omega)+d_{23}(\omega,h_1,h_2,\dots,h_{n-1})\gamma_1(\omega)+\\
+d_{24}(\omega,h_1,h_2,\dots,h_{n-1})\delta_1(\omega)\big)
\end{gathered}
$$

The second equation can be rewritten as

$$
\begin{aligned}
(60)\quad &\Big(E_n\omega d_{31}-\frac{s(\omega)}{k_{1n}^2}d_{21}\Big)\alpha_1(\omega)+\Big(E_n\omega d_{32}-\frac{s(\omega)}{k_{1n}^2}d_{22}\Big)\beta_1(\omega)+\\
&+\Big(E_n\omega d_{33}-\frac{s(\omega)}{k_{1n}^2}d_{23}\Big)\gamma_1(\omega)+\Big(E_n\omega d_{34}-\frac{s(\omega)}{k_{1n}^2}d_{24}\Big)\delta_1(\omega)=0
\end{aligned}
$$

If we put together equation (59) and (60) we have a system of two linear equations with four unknowns α_1, β_1, γ_1 and δ_1. In order to obtain the full problem we also use the two boundary conditions on the interface Γ_3:

$$
\begin{aligned}
\hat{\sigma}_y(\omega,h) &= \hat{l}(\omega)=iE\omega\gamma_1(\omega)\\
\hat{\tau}_{xy}(\omega,h) &= \hat{g}(\omega)=\mu\omega\delta_1(\omega).
\end{aligned}
$$

In other words we know $\gamma_1(\omega)$ and $\delta_1(\omega)$. After putting these values in the system (59), (60) we will be able to determine the two unknown functions $\alpha_1(\omega)$ and $\beta_1(\omega)$. So, we will have the full vector $\overrightarrow{\alpha}_1(\omega)$ and with the recurrence relation (58) we can find the deformations of each layer in our multilayer pack.

It would be great if we could calculate the elements d_{ij} but in practice it is almost impossible[9] and we will use a less obvious way.

We repeat one more time that we have the two equations (59), (60) with unknowns $\alpha_1(\omega)$ and $\beta_1(\omega)$. Suppose we have solved these equations:

$$
\begin{aligned}
\alpha_1(\omega) &= -A_n(\omega,h_1,\dots,h_{n-1})\gamma_1(\omega)-C_n(\omega,h_1,\dots,h_{n-1})\delta_1(\omega)\\
\beta_1(\omega) &= -B_n(\omega,h_1,\dots,h_{n-1})\gamma_1(\omega)-D_n(\omega,h_1,\dots,h_{n-1})\delta_1(\omega)
\end{aligned}
$$

The functions A_n, B_n, C_n, D_n are named Lamzyuk-Privarnikov functions because this method was proposed by A. K. Privarnikov and developed later by V. D. Lamzyuk.[10]

[9] We have to calculate the products of matrices. Analytically it is awkward. Numerically it is not interesting.

[10] Many thanks to my teacher V.D. Lamzyuk who learned me this technic and not only this.

Our main problem is to obtain these four functions. We construct recurrence relations for them.

Let us assume that we have constructed these functions for a multilayer pack with $(n-1)$ layers. The numbering of layers starts from 2 to n:

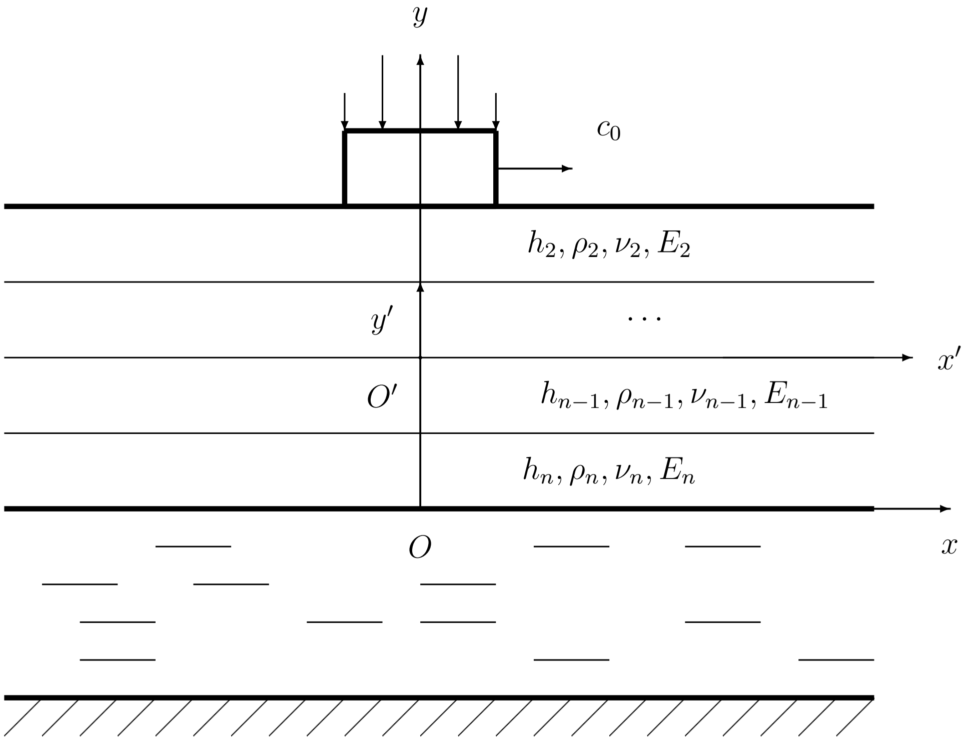

FIGURE 4. Moving load on the multilayer plate. Illustration for special layers numeration

By definition of the Lamzyuk-Privarnikov functions we have:

$$\begin{aligned}\alpha_2(\omega) &= -A_{n-1}(\omega, h_2, \dots, h_{n-1})\gamma_2(\omega) - C_{n-1}(\omega, h_2, \dots, h_{n-1})\delta_2(\omega) \\ \beta_2(\omega) &= -B_{n-1}(\omega, h_2, \dots, h_{n-1})\gamma_2(\omega) - D_{n-1}(\omega, h_2, \dots, h_{n-1})\delta_2(\omega)\end{aligned}$$

Next we put one more layer on top of the upper boundary of the previous multilayer pack (Figure 4). This new layer is labelled 1. So, we obtain the situation illustrated on Figure 3.

Earlier we established the recurrence relation (58) between $\overrightarrow{\alpha}_j$ and $\overrightarrow{\alpha}_{j+1}$. Now we use it with $j = 1$:

$$\overrightarrow{\alpha}_2(\omega) = I_2^{-1} I_1 H_1(-h_1)\overrightarrow{\alpha}_1(\omega) = B(\omega, h_1)\overrightarrow{\alpha}_1(\omega),$$

where we introduced the notation $B(\omega, h) = (b_{ij})_{1\le i,j\le 4} := I_2^{-1} I_1 H_1(-h)$. Recall the definition of the diagonal matrix $I_1 = \operatorname{diag}\left\{1, \frac{i}{k_{11}^2}, iE_1\omega, \mu_1\omega\right\}$

and $I_2^{-1} = \operatorname{diag}\left\{1, \frac{k_{12}^2}{i}, \frac{1}{iE_2\omega}, \frac{1}{\mu_2\omega}\right\}$. It is not difficult to calculate explicitly the coefficients (b_{ij}):

$$b_{11}(\omega, h) = \frac{B_{11}\cosh(k_{21}\omega h) - B_{21}\cosh(k_{11}\omega h)}{B_{11} - B_{21}},$$

$$b_{12}(\omega, h) = \frac{A_1\sinh(k_{21}\omega h) - 2k_{21}\sinh(k_{11}\omega h)}{k_{11}k_{21}(2 - A_1k_{11})},$$

$$b_{13}(\omega, h) = \frac{\cosh(k_{11}\omega h) - \cosh(k_{21}\omega h)}{B_{11} - B_{21}},$$

$$b_{14}(\omega, h) = \frac{k_{11}k_{21}\sinh(k_{11}\omega h) - \sinh(k_{21}\omega h)}{k_{21}(2 - A_1k_{11})},$$

$$b_{21}(\omega, h) = \frac{k_{12}^2}{k_{11}}\frac{B_{21}\sinh(k_{11}\omega h) - B_{11}k_{11}k_{21}\sinh(k_{21}\omega h)}{B_{11} - B_{21}},$$

$$b_{22}(\omega, h) = \frac{k_{12}^2}{k_{11}^2}\frac{2\cosh(k_{11}\omega h) - A_1k_{11}\cosh(k_{21}\omega h)}{2 - A_1k_{11}},$$

$$b_{23}(\omega, h) = \frac{k_{12}^2}{k_{11}}\frac{k_{11}k_{21}\sinh(k_{21}\omega h) - \sinh(k_{11}\omega h)}{B_{11} - B_{21}},$$

$$b_{24}(\omega, h) = k_{12}^2\frac{\cosh(k_{21}\omega h) - \cosh(k_{11}\omega h)}{2 - A_1k_{11}},$$

$$b_{31}(\omega, h) = \frac{E_1}{E_2}B_{11}B_{21}\frac{\cosh(k_{21}\omega h) - \cosh(k_{11}\omega h)}{B_{11} - B_{21}},$$

$$b_{32}(\omega, h) = \frac{E_1}{E_2}\frac{A_1B_{21}\sinh(k_{21}\omega h) - 2B_{11}k_{21}\sinh(k_{11}\omega h)}{k_{11}k_{21}(2 - A_1k_{11})},$$

$$b_{33}(\omega, h) = \frac{E_1}{E_2}\frac{B_{11}\cosh(k_{11}\omega h) - B_{21}\cosh(k_{21}\omega h)}{B_{11} - B_{21}},$$

$$b_{34}(\omega, h) = \frac{E_1}{E_2}\frac{B_{11}k_{11}k_{21}\sinh(k_{11}\omega h) - B_{21}\sinh(k_{21}\omega h)}{k_{21}(2 - A_1k_{11})},$$

$$b_{41}(\omega, h) = \frac{\mu_1}{\mu_2}\frac{A_1B_{21}\sinh(k_{11}\omega h) - 2B_{11}k_{21}\sinh(k_{21}\omega h)}{B_{11} - B_{21}},$$

$$b_{42}(\omega, h) = \frac{\mu_1}{\mu_2}2A_1\frac{\cosh(k_{11}\omega h) - \cosh(k_{21}\omega h)}{k_{11}(2 - A_1k_{11})},$$

$$b_{43}(\omega, h) = \frac{\mu_1}{\mu_2}\frac{2k_{21}\sinh(k_{21}\omega h) - A_1\sinh(k_{11}\omega h)}{B_{11} - B_{21}},$$

$$b_{44}(\omega, h) = \frac{\mu_1}{\mu_2}\frac{2\cosh(k_{21}\omega h) - A_1k_{11}\cosh(k_{11}\omega h)}{2 - A_1k_{11}}.$$

The matrix B provides us a linear relation between $\overrightarrow{\alpha}_2(\omega)$ and $\overrightarrow{\alpha}_1(\omega)$. The extended formulae are

$$\begin{aligned}
\alpha_2(\omega) &= b_{11}(\omega,h)\alpha_1(\omega)+b_{12}(\omega,h)\beta_1(\omega)+b_{13}(\omega,h)\gamma_1(\omega)+b_{14}(\omega,h)\delta_1(\omega),\\
\beta_2(\omega) &= b_{21}(\omega,h)\alpha_1(\omega)+b_{22}(\omega,h)\beta_1(\omega)+b_{23}(\omega,h)\gamma_1(\omega)+b_{24}(\omega,h)\delta_1(\omega),\\
\gamma_2(\omega) &= b_{31}(\omega,h)\alpha_1(\omega)+b_{32}(\omega,h)\beta_1(\omega)+b_{33}(\omega,h)\gamma_1(\omega)+b_{34}(\omega,h)\delta_1(\omega),\\
\delta_2(\omega) &= b_{41}(\omega,h)\alpha_1(\omega)+b_{42}(\omega,h)\beta_1(\omega)+b_{43}(\omega,h)\gamma_1(\omega)+b_{44}(\omega,h)\delta_1(\omega).
\end{aligned}$$

In the definition of Lamzyuk-Privarnikov functions for $(n-1)$ layers, we have

$$\begin{aligned}
\alpha_2(\omega) &= -A_{n-1}(\omega,h_2,\ldots,h_{n-1})\gamma_2(\omega)-C_{n-1}(\omega,h_2,\ldots,h_{n-1})\delta_2(\omega),\\
\beta_2(\omega) &= -B_{n-1}(\omega,h_2,\ldots,h_{n-1})\gamma_2(\omega)-D_{n-1}(\omega,h_2,\ldots,h_{n-1})\delta_2(\omega).
\end{aligned}$$

One can easily obtain two identities:

$$(b_{11}+A_{n-1}b_{31}+C_{n-1}b_{41})\alpha_1(\omega)+(b_{12}+A_{n-1}b_{32}+C_{n-1}b_{42})\beta_1(\omega)+ \\ +(b_{13}+A_{n-1}b_{33}+C_{n-1}b_{43})\gamma_1(\omega)+(b_{14}+A_{n-1}b_{34}+C_{n-1}b_{44})\delta_1(\omega)=0,$$

$$(b_{21}+B_{n-1}b_{31}+D_{n-1}b_{41})\alpha_1(\omega)+(b_{22}+B_{n-1}b_{32}+D_{n-1}b_{42})\beta_1(\omega)+ \\ +(b_{23}+B_{n-1}b_{33}+D_{n-1}b_{43})\gamma_1(\omega)+(b_{24}+B_{n-1}b_{34}+D_{n-1}b_{44})\delta_1(\omega)=0.$$

If we remember that in these identities the unknowns are $\alpha_1(\omega)$ and $\beta_1(\omega)$ we can easily obtain a linear system of two equations:

$$(b_{11}+A_{n-1}b_{31}+C_{n-1}b_{41})\alpha_1(\omega)+(b_{12}+A_{n-1}b_{32}+C_{n-1}b_{42})\beta_1(\omega)= \\ =-(b_{13}+A_{n-1}b_{33}+C_{n-1}b_{43})\gamma_1(\omega)-(b_{14}+A_{n-1}b_{34}+C_{n-1}b_{44})\delta_1(\omega),$$

$$(b_{21}+B_{n-1}b_{31}+D_{n-1}b_{41})\alpha_1(\omega)+(b_{22}+B_{n-1}b_{32}+D_{n-1}b_{42})\beta_1(\omega)= \\ =-(b_{23}+B_{n-1}b_{33}+D_{n-1}b_{43})\gamma_1(\omega)-(b_{24}+B_{n-1}b_{34}+D_{n-1}b_{44})\delta_1(\omega).$$

The solution of this system is

$$\alpha_1(\omega) = -\frac{Y_n(\omega)\gamma_1(\omega)+Q_n(\omega)\delta_1(\omega)}{R_n(\omega)}, \tag{61}$$

$$\beta_1(\omega) = -\frac{X_n(\omega)\gamma_1(\omega)+Z_n(\omega)\delta_1(\omega)}{R_n(\omega)}, \tag{62}$$

where the functions $Y_n(\omega)$, $Q_n(\omega)$, $X_n(\omega)$, $Z_n(\omega)$, $R_n(\omega)$ are determined by Cramer's rule:

$$R_n(\omega)=\begin{vmatrix} b_{11}+A_{n-1}b_{31}+C_{n-1}b_{41} & b_{12}+A_{n-1}b_{32}+C_{n-1}b_{42}\\ b_{21}+B_{n-1}b_{31}+D_{n-1}b_{41} & b_{22}+B_{n-1}b_{32}+D_{n-1}b_{42}\end{vmatrix},$$

$$Y_n(\omega)=\begin{vmatrix} b_{13}+A_{n-1}b_{33}+C_{n-1}b_{43} & b_{12}+A_{n-1}b_{32}+C_{n-1}b_{42}\\ b_{23}+B_{n-1}b_{33}+D_{n-1}b_{43} & b_{22}+B_{n-1}b_{32}+D_{n-1}b_{42}\end{vmatrix},$$

$$Q_n(\omega)=\begin{vmatrix} b_{14}+A_{n-1}b_{34}+C_{n-1}b_{44} & b_{12}+A_{n-1}b_{32}+C_{n-1}b_{42}\\ b_{24}+B_{n-1}b_{34}+D_{n-1}b_{44} & b_{22}+B_{n-1}b_{32}+D_{n-1}b_{42}\end{vmatrix},$$

$$X_n(\omega) = \begin{vmatrix} b_{11} + A_{n-1}b_{31} + C_{n-1}b_{41} & b_{13} + A_{n-1}b_{33} + C_{n-1}b_{43} \\ b_{21} + B_{n-1}b_{31} + D_{n-1}b_{41} & b_{23} + B_{n-1}b_{33} + D_{n-1}b_{43} \end{vmatrix},$$

$$Z_n(\omega) = \begin{vmatrix} b_{11} + A_{n-1}b_{31} + C_{n-1}b_{41} & b_{14} + A_{n-1}b_{34} + C_{n-1}b_{44} \\ b_{21} + B_{n-1}b_{31} + D_{n-1}b_{41} & b_{24} + B_{n-1}b_{34} + D_{n-1}b_{44} \end{vmatrix}.$$

If the reader compares the identities (61), (62) with the definition of Lamzyuk-Privarnikov functions

$$\begin{aligned} \alpha_1(\omega) &= -A_n(\omega, h_1, \ldots, h_{n-1})\gamma_1(\omega) - C_n(\omega, h_1, \ldots, h_{n-1})\delta_1(\omega) \\ \beta_1(\omega) &= -B_n(\omega, h_1, \ldots, h_{n-1})\gamma_1(\omega) - D_n(\omega, h_1, \ldots, h_{n-1})\delta_1(\omega) \end{aligned}$$

it is easy to conclude that

$$A_n(\omega) = \frac{Y_n(\omega)}{R_n(\omega)}, \qquad B_n(\omega) = \frac{X_n(\omega)}{R_n(\omega)},$$

$$C_n(\omega) = \frac{Q_n(\omega)}{R_n(\omega)}, \qquad D_n(\omega) = \frac{Z_n(\omega)}{R_n(\omega)}.$$

We have calculated analytical expressions of the functions $Y_n(\omega)$, $Q_n(\omega)$, $X_n(\omega)$, $Z_n(\omega)$, $R_n(\omega)$. Here they are:

$$Y_n(\omega) = g_{11}(\omega) + g_{12}(\omega)A_{n-1}(\omega) + g_{13}(\omega)B_{n-1}(\omega) + \\ + g_{14}(\omega)C_{n-1}(\omega) + g_{15}(\omega)D_{n-1}(\omega) + g_{16}(\omega)\Delta_{n-1}(\omega),$$

$$X_n(\omega) = g_{21}(\omega) + g_{22}(\omega)A_{n-1}(\omega) + g_{23}(\omega)B_{n-1}(\omega) + \\ + g_{24}(\omega)C_{n-1}(\omega) + g_{25}(\omega)D_{n-1}(\omega) + g_{26}(\omega)\Delta_{n-1}(\omega),$$

$$Q_n(\omega) = g_{31}(\omega) + g_{32}(\omega)A_{n-1}(\omega) + g_{33}(\omega)B_{n-1}(\omega) + \\ + g_{34}(\omega)C_{n-1}(\omega) + g_{35}(\omega)D_{n-1}(\omega) + g_{36}(\omega)\Delta_{n-1}(\omega),$$

$$Z_n(\omega) = g_{41}(\omega) + g_{42}(\omega)A_{n-1}(\omega) + g_{43}(\omega)B_{n-1}(\omega) + \\ + g_{44}(\omega)C_{n-1}(\omega) + g_{45}(\omega)D_{n-1}(\omega) + g_{46}(\omega)\Delta_{n-1}(\omega),$$

$$R_n(\omega) = g_{51}(\omega) + g_{52}(\omega)A_{n-1}(\omega) + g_{53}(\omega)B_{n-1}(\omega) + \\ + g_{54}(\omega)C_{n-1}(\omega) + g_{55}(\omega)D_{n-1}(\omega) + g_{56}(\omega)\Delta_{n-1}(\omega),$$

where

$$\Delta_{n-1}(\omega) = \begin{vmatrix} A_{n-1}(\omega) & B_{n-1}(\omega) \\ C_{n-1}(\omega) & D_{n-1}(\omega) \end{vmatrix} = A_{n-1}(\omega)D_{n-1}(\omega) - B_{n-1}(\omega)C_{n-1}(\omega).$$

The expressions of the coefficients $g_{ij}(\omega)$, $1 \le i \le 5$, $1 \le j \le 6$ are:

$$g_{11}(\omega) = b_{13}b_{22} - b_{12}b_{23} = \frac{k_{12}^2}{k_1 k_2 (2 - Ak_1)(B_1 - B_2)} \Big(k_2(2 + Ak_1) \times \\ \times (1 - \cosh(k_1\omega h)\cosh(k_2\omega h)) + (A + 2k_1k_2^2)\sinh(k_1\omega h)\sinh(k_2\omega h) \Big),$$

$$g_{12}(\omega) = b_{33}b_{22} - b_{23}b_{32} = \frac{E_1 k_{12}^2}{E_2 k_1 k_2 (2 - Ak_1)(B_1 - B_2)} \times$$
$$\Big((2B_1 + AB_2 k_1)k_2 - k_2(AB_1 k_1 + 2B_2)\cosh(k_1\omega h)\cosh(k_2\omega h) +$$
$$+ (AB_2 + 2B_1 k_1 k_2^2)\sinh(k_1\omega h)\sinh(k_2\omega h), \Big)$$

$$g_{13}(\omega) = b_{13}b_{32} - b_{12}b_{33} = \frac{E_1}{E_2 k_2 (2 - Ak_1)} \times$$
$$\Big(2k_2 \sinh(k_1\omega h)\cosh(k_2\omega h) - A\cosh(k_1\omega h)\sinh(k_2\omega h)\Big)$$

$$g_{14}(\omega) = b_{43}b_{22} - b_{42}b_{23} = \frac{\mu_1 k_{12}^2}{\mu_2 k_1 (B_1 - B_2)} \times$$
$$\Big(2k_2 \cosh(k_1\omega h)\sinh(k_2\omega h) - A\sinh(k_1\omega h)\cosh(k_2\omega h)\Big),$$

$$g_{15}(\omega) = b_{13}b_{42} - b_{12}b_{43} = \frac{\mu_1}{\mu_2 k_2 (2 - Ak_1)(B_1 - B_2)} \times$$
$$\Big(4Ak_2(1 - \cosh(k_1\omega h)\cosh(k_2\omega h)) +$$
$$+ (A^2 + 4k_2^2)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{16}(\omega) = b_{33}b_{42} - b_{32}b_{43} = \Big(2Ak_2(B_1 + B_2)\times$$
$$(1 - \cosh(k_1\omega h)\cosh(k_2\omega h)) + (A^2 B_2 + 4k_2^2 B_1)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{21}(\omega) = b_{11}b_{23} - b_{13}b_{21} = \frac{k_{12}^2}{B_1 - B_2}\Big(k_1 k_2 \cosh(k_1\omega h)\sinh(k_2\omega h) -$$
$$- \sinh(k_1\omega h)\cosh(k_2\omega h)\Big),$$

$$g_{22}(\omega) = b_{31}b_{23} - b_{33}b_{21} = \frac{E_1 k_{12}^2}{E_2 (B_1 - B_2)} \times$$
$$\times \Big(B_1 k_1 k_2 \cosh(k_1\omega h)\sinh(k_2\omega h) - B_2 \sinh(k_1\omega h)\cosh(k_2\omega h)\Big),$$

$$g_{23}(\omega) = b_{11}b_{33} - b_{13}b_{31} = \frac{E_1 k_1}{E_2}\cosh(k_1\omega h)\cosh(k_2\omega h),$$

$$g_{24}(\omega) = b_{41}b_{23} - b_{43}b_{21} = \frac{\mu_1 k_{12}^2 k_2 (2 - Ak_1)}{\mu_2 (B_1 - B_2)}\sinh(k_1\omega h)\sinh(k_2\omega h),$$

$$g_{25}(\omega) = b_{11}b_{43} - b_{13}b_{41} = \frac{\mu_1 k_1}{\mu_2(B_1 - B_2)}\Big(2k_2 \cosh(k_1\omega h)\sinh(k_2\omega h) - \\ - A\sinh(k_1\omega h)\cosh(k_2\omega h)\Big),$$

$$g_{26}(\omega) = b_{31}b_{43} - b_{33}b_{41} = \frac{E_1\mu_1 k_1}{E_2\mu_2(B_1 - B_2)}\times \\ \times \Big(2B_1k_2 \cosh(k_1\omega h)\sinh(k_2\omega h) - AB_2\sinh(k_1\omega h)\cosh(k_2\omega h)\Big),$$

$$g_{31}(\omega) = b_{14}b_{22} - b_{12}b_{24} = \frac{k_{12}^2}{k_1k_2(2 - Ak_1)}\times \\ \times \Big(k_1k_2\sinh(k_1\omega h)\cosh(k_2\omega h) - \cosh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{32}(\omega) = b_{34}b_{22} - b_{32}b_{24} = \frac{E_1 k_{12}^2}{E_2k_1k_2(2 - Ak_1)}\times \\ \times \Big(B_1k_1k_2\sinh(k_1\omega h)\cosh(k_2\omega h) - B_2\cosh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{33}(\omega) = b_{14}b_{32} - b_{12}b_{34} = \frac{E_1(B_1 - B_2)}{E_2k_2(2 - Ak_1)}\sinh(k_1\omega h)\sinh(k_2\omega h),$$

$$g_{34}(\omega) = b_{44}b_{22} - b_{42}b_{24} = \frac{\mu_1 k_{12}^2}{\mu_2 k_{11}}\cosh(k_1\omega h)\cosh(k_2\omega h),$$

$$g_{35}(\omega) = b_{14}b_{42} - b_{12}b_{44} = \frac{\mu_1}{\mu_2k_2(2 - Ak_1)}\times \\ \times \Big(2k_2\sinh(k_1\omega h)\cosh(k_2\omega h) - A\cosh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{36}(\omega) = b_{34}b_{42} - b_{32}b_{44} = \frac{E_1\mu_1}{E_2\mu_2k_2(2 - Ak_1)}\times \\ \times \Big(2B_1k_2\sinh(k_1\omega h)\cosh(k_2\omega h) - AB_2\cosh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{41}(\omega) = b_{11}b_{24} - b_{14}b_{21} = \frac{k_{12}^2}{k_2(B_1 - B_2)(2 - Ak_1)}\times \\ \times \Big(k_1k_2(B_1 + B_2)(1 - \cosh(k_1\omega h)\cosh(k_2\omega h)) + \\ + (B_1k_1^2k_2^2 + B_2)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{42}(\omega) = b_{31}b_{24} - b_{34}b_{21} = \frac{E_1 k_{12}^2}{E_2 k_2 (2 - Ak_1)(B_1 - B_2)} \times$$
$$\times \Big(2B_1B_2k_1k_2(1 - \cosh(k_1\omega h)\cosh(k_2\omega h)) +$$
$$+ (B_1^2k_1^2k_2^2 + B_2^2)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{43}(\omega) = b_{11}b_{34} - b_{14}b_{31} = \frac{E_1 k_1}{E_2 k_2 (2 - Ak_1)} \times$$
$$\times \Big(B_1k_1k_2\sinh(k_1\omega h)\cosh(k_2\omega h) - B_2\cosh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{44}(\omega) = b_{41}b_{24} - b_{44}b_{21} = \frac{\mu_1 k_{12}^2}{\mu_2 (B_1 - B_2)} \times$$
$$\times \Big(B_1k_1k_2\cosh(k_1\omega h)\sinh(k_2\omega h) - B_2\sinh(k_1\omega h)\cosh(k_2\omega h)\Big),$$

$$g_{45}(\omega) = b_{11}b_{44} - b_{14}b_{41} = \frac{\mu_1 k_{11}}{\mu_2 k_{21} (2 - Ak_1)(B_1 - B_2)} \times$$
$$\times \Big((2B_1k_2 + AB_2k_1k_2) - (2B_2k_2 + AB_1k_1k_2)\cosh(k_1\omega h)\cosh(k_2\omega h) +$$
$$+ (2B_1k_1k_2^2 + AB_2)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{46}(\omega) = b_{31}b_{44} - b_{34}b_{41} = \frac{E_1\mu_1 k_1}{\mu_2 E_2 (2 - Ak_1)(B_1 - B_2)} \times$$
$$\times \Big(B_1B_2k_2(2 + Ak_1)(1 - \cosh(k_1\omega h)\cosh(k_2\omega h)) +$$
$$+ (AB_2^2 + 2B_1^2k_1k_2^2)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{51}(\omega) = b_{11}b_{22} - b_{12}b_{21} = \frac{k_{12}^2}{k_1 k_2 (2 - Ak_1)(B_1 - B_2)} \times$$
$$\Big((2B_1k_2 + AB_2k_1k_2)\cosh(k_1\omega h)\cosh(k_2\omega h) -$$
$$- (2k_2B_2 + AB_1k_1k_2) - (AB_2 + 2k_1k_2^2B_1)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{52}(\omega) = b_{31}b_{22} - b_{32}b_{21} = \frac{E_1 k_{12}^2}{E_2 k_1 k_2 (2 - Ak_1)(B_1 - B_2)} \times \Big(B_1 B_2 k_2 (2 + Ak_1)[\cosh(k_1\omega h)\cosh(k_2\omega h) - 1] - (AB_{21}^2 + 2B_1^2 k_1 k_2^2)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{53}(\omega) = b_{11}b_{32} - b_{12}b_{31} = \frac{E_1}{E_2 k_2 (2 - Ak_1)} \times \Big(AB_2 \cosh(k_1\omega h)\sinh(k_2\omega h) - 2B_1 k_2 \sinh(k_1\omega h)\cosh(k_2\omega h)\Big),$$

$$g_{54}(\omega) = b_{41}b_{22} - b_{42}b_{21} = \frac{\mu_1 k_{12}^2}{\mu_2 k_1 (B_1 - B_2)} \times \Big(AB_2 \sinh(k_1\omega h)\cosh(k_2\omega h) - 2B_1 k_2 \cosh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{55}(\omega) = b_{11}b_{42} - b_{12}b_{41} = \frac{\mu_1}{\mu_2 k_2 (2 - Ak_1)(B_1 - B_2)} \Big(2Ak_2(B_1 + B_2) \times [\cosh(k_1\omega h)\cosh(k_2\omega h) - 1] - (A^2 B_2 + 4k_2^2 B_1)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

$$g_{56}(\omega) = b_{31}b_{42} - b_{32}b_{41} = \frac{E_1 \mu_1}{E_2 \mu_2 k_2 (2 - Ak_1)(B_1 - B_2)} \Big(4AB_1 B_2 k_2 \times (\cosh(k_1\omega h)\cosh(k_2\omega h) - 1) - (A^2 B_2^2 + 4B_1^2 k_2^2)\sinh(k_1\omega h)\sinh(k_2\omega h)\Big),$$

Here we constructed the Lamzyuk-Privarnikov functions for multi-layer pack. These functions depend only on mechanical properties and geometrical characteristics of the pack and do not depend on the boundary conditions. It means that once constructed these functions can be used to solve different problems. That is why we find this method amazing and very useful.

11. Inhomogeneous layer limit

In this section we take the limit as the number of layers tends to infinity and the maximum thickness tends to zero with the constraint

$$\sum_{i=1}^{n} h_i = h.$$

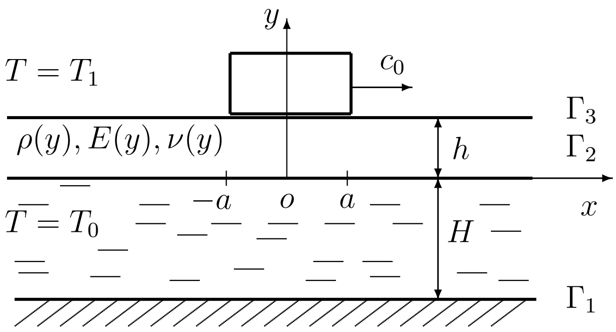

FIGURE 5. Moving load on the inhomogeneous layer.

Let us introduce continuous notation. So far we have considered the piecewise constant function $\overrightarrow{\alpha}(\omega, y)$ which is defined by

$$\overrightarrow{\alpha}(\omega, y) = \overrightarrow{\alpha}_k(\omega), \qquad \sum_{i=0}^{n-k-1} h_{n-i} \le y < \sum_{i=0}^{n-k} h_{n-i}, k = 1, \dots, n-1.$$

To simplify the notation we assume that all layers have the same thickness. In other words we have

$$\alpha_k(\omega) \equiv \alpha(\omega, y), \qquad \alpha_{k+1}(\omega) \equiv \alpha(\omega, y-h)$$

$$\beta_k(\omega) \equiv \beta(\omega, y), \qquad \beta_{k+1}(\omega) \equiv \beta(\omega, y-h)$$

$$\delta_k(\omega) \equiv \delta(\omega, y), \qquad \delta_{k+1}(\omega) \equiv \delta(\omega, y-h)$$

$$\gamma_k(\omega) \equiv \gamma(\omega, y), \qquad \gamma_{k+1}(\omega) \equiv \gamma(\omega, y-h)$$

In the previous section we obtained the recurrence relation (58) for the vector $\overrightarrow{\alpha}_k(\omega)$. This relation gives the connection between two vectors $\overrightarrow{\alpha}(\omega)$ for consecutive layers in the multilayer pack. We have also established that knowing the vector $\overrightarrow{\alpha}$ is equivalent to knowing the solution to our problem, since

$$\text{(63)} \qquad \begin{pmatrix} u(x,y) \\ v(x,y) \\ \sigma_y(x,y) \\ \tau_{xy}(x,y) \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} IH(\omega, y)\overrightarrow{\alpha}(\omega, y) e^{-i\omega x} d\omega.$$

Let

$$B_j(\omega, h_k) = (b_{ij})_{1 \le i,j \le 4} := I_{k+1}^{-1} I_k H_k(\omega, -h_k).$$

We can rewrite the recurrence relation (58) in the form

$$\begin{Vmatrix} \alpha_{k+1}(\omega) \\ \beta_{k+1}(\omega) \\ \gamma_{k+1}(\omega) \\ \delta_{k+1}(\omega) \end{Vmatrix} = \begin{Vmatrix} b_{11}(\omega,h) & b_{12}(\omega,h) & b_{13}(\omega,h) & b_{14}(\omega,h) \\ b_{21}(\omega,h) & b_{22}(\omega,h) & b_{23}(\omega,h) & b_{24}(\omega,h) \\ b_{31}(\omega,h) & b_{32}(\omega,h) & b_{33}(\omega,h) & b_{34}(\omega,h) \\ b_{41}(\omega,h) & b_{42}(\omega,h) & b_{43}(\omega,h) & b_{44}(\omega,h) \end{Vmatrix} \cdot \begin{Vmatrix} \alpha_k(\omega) \\ \beta_k(\omega) \\ \gamma_k(\omega) \\ \delta_k(\omega) \end{Vmatrix}$$

or in continous notations:

$$\left\|\begin{matrix}\alpha(\omega,y-h)\\ \beta(\omega,y-h)\\ \gamma(\omega,y-h)\\ \delta(\omega,y-h)\end{matrix}\right\| = \left\|\begin{matrix} b_{11}(\omega,h) & b_{12}(\omega,h) & b_{13}(\omega,h) & b_{14}(\omega,h)\\ b_{21}(\omega,h) & b_{22}(\omega,h) & b_{23}(\omega,h) & b_{24}(\omega,h)\\ b_{31}(\omega,h) & b_{32}(\omega,h) & b_{33}(\omega,h) & b_{34}(\omega,h)\\ b_{41}(\omega,h) & b_{42}(\omega,h) & b_{43}(\omega,h) & b_{44}(\omega,h)\end{matrix}\right\| \cdot \left\|\begin{matrix}\alpha(\omega,y)\\ \beta(\omega,y)\\ \gamma(\omega,y)\\ \delta(\omega,y)\end{matrix}\right\|$$

In order to obtain a system of differential equations we subtract from both parts the vector $\overrightarrow{\alpha}(\omega,y)$ and divide then by $-h$:

$$\frac{\overrightarrow{\alpha}(\omega,y-h)-\overrightarrow{\alpha}(\omega,y)}{-h} = \frac{B_k(\omega,h)-Id}{-h}\overrightarrow{\alpha}(\omega,y)$$

or in expanded form:

$$\left\|\begin{matrix}\frac{\alpha(\omega,y-h)-\alpha(\omega,y)}{-h}\\ \frac{\beta(\omega,y-h)-\beta(\omega,y)}{-h}\\ \frac{\gamma(\omega,y-h)-\gamma(\omega,y)}{-h}\\ \frac{\delta(\omega,y-h)-\delta(\omega,y)}{-h}\end{matrix}\right\| = \left\|\begin{matrix}\frac{b_{11}(\omega,y)-1}{-h} & \frac{b_{12}(\omega,y)}{-h} & \frac{b_{13}(\omega,y)}{-h} & \frac{b_{14}(\omega,y)}{-h}\\ \frac{b_{21}(\omega,y)}{-h} & \frac{b_{22}(\omega,y)-1}{-h} & \frac{b_{23}(\omega,y)}{-h} & \frac{b_{24}(\omega,y)}{-h}\\ \frac{b_{31}(\omega,y)}{-h} & \frac{b_{32}(\omega,y)}{-h} & \frac{b_{33}(\omega,y)-1}{-h} & \frac{b_{34}(\omega,y)}{-h}\\ \frac{b_{41}(\omega,y)}{-h} & \frac{b_{42}(\omega,y)}{-h} & \frac{b_{43}(\omega,y)}{-h} & \frac{b_{44}(\omega,y)-1}{-h}\end{matrix}\right\| \cdot \left\|\begin{matrix}\alpha(\omega,y)\\ \beta(\omega,y)\\ \gamma(\omega,y)\\ \delta(\omega,y)\end{matrix}\right\|$$

Let $h \to 0$. One obtains a system of linear differential equations:

$$\left\|\begin{matrix}\frac{d\alpha(\omega,y)}{dy}\\ \frac{d\beta(\omega,y)}{dy}\\ \frac{d\gamma(\omega,y)}{dy}\\ \frac{d\delta(\omega,y)}{dy}\end{matrix}\right\| = \left\|\begin{matrix}\lim\limits_{h\to0}\frac{b_{11}(\omega,y)-1}{-h} & \lim\limits_{h\to0}\frac{b_{12}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{13}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{14}(\omega,y)}{-h}\\ \lim\limits_{h\to0}\frac{b_{21}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{22}(\omega,y)-1}{-h} & \lim\limits_{h\to0}\frac{b_{23}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{24}(\omega,y)}{-h}\\ \lim\limits_{h\to0}\frac{b_{31}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{32}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{33}(\omega,y)-1}{-h} & \lim\limits_{h\to0}\frac{b_{34}(\omega,y)}{-h}\\ \lim\limits_{h\to0}\frac{b_{41}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{42}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{43}(\omega,y)}{-h} & \lim\limits_{h\to0}\frac{b_{44}(\omega,y)-1}{-h}\end{matrix}\right\| \cdot \left\|\begin{matrix}\alpha(\omega,y)\\ \beta(\omega,y)\\ \gamma(\omega,y)\\ \delta(\omega,y)\end{matrix}\right\|$$

or, in matrix form,

$$\frac{d\overrightarrow{\alpha}}{dy} = A\overrightarrow{\alpha}(\omega,y).$$

Now we have to calculate the matrix A.

$$\lim_{h\to0}\frac{b_{11}-1}{-h} = \lim_{h\to0}\frac{b_{13}}{-h} = 0,$$

$$\lim_{h\to0}\frac{b_{12}}{-h} = \frac{2k_1-A}{k_1(2-Ak_1)}\omega,$$

$$\lim_{h\to0}\frac{b_{14}}{-h} = \frac{1-k_1^2}{2-Ak_1}\omega.$$

So, we obtain the first differential equation:

$$\frac{d\alpha}{dy} = \frac{2k_1-A}{k_1(2-Ak_1)}\omega\beta(\omega,y) + \frac{1-k_1^2}{2-Ak_1}\omega\delta(\omega,y)$$

or after simplification

$$\frac{d\alpha}{dy} = -\frac{\omega}{k_1^2}\beta(\omega,y) + \omega\delta(\omega,y) \tag{64}$$

We consider the second equation:

$$\lim_{h\to0}\frac{b_{21}}{-h} = \omega k_1^2\frac{B_1k_2^2-B_2}{B_1-B_2},$$

$$\lim_{h\to 0}\frac{b_{22}-1}{-h}=2\frac{d}{dy}\ln k_1(y),$$

$$\lim_{h\to 0}\frac{b_{23}}{-h}=\omega k_1^2\frac{1-k_1^2}{B_1-B_2},$$

$$\lim_{h\to 0}\frac{b_{24}}{-h}=0.$$

Therefore

$$(65)\quad \frac{d\beta}{dy}=\omega k_1^2\frac{B_1k_2^2-B_2}{B_1-B_2}\alpha(\omega,y)+ \\ +2\frac{d}{dy}\ln k_1(y)\beta(\omega,y)+\omega k_1^2\frac{1-k_1^2}{B_1-B_2}\gamma(\omega,y)$$

Third equation:

$$\lim_{h\to 0}\frac{b_{31}}{-h}=0,$$

$$\lim_{h\to 0}\frac{b_{32}}{-h}=\omega\frac{2B_1k_1-AB_2}{k_1(2-Ak_1)},$$

$$\lim_{h\to 0}\frac{b_{33}-1}{-h}=-\frac{d}{dy}\ln E(y),$$

$$\lim_{h\to 0}\frac{b_{34}}{-h}=\omega\frac{B_2-B_1k_1^2}{2-Ak_1}.$$

$$(66)\quad \frac{d\gamma}{dy}=\omega\frac{2B_1k_1-AB_2}{k_1(2-Ak_1)}\beta(\omega,y)-\frac{d}{dy}\ln E(y)\gamma(\omega,y)+ \\ +\omega\frac{B_2-B_1k_1^2}{2-Ak_1}\delta(\omega,y)$$

Fourth equation:

$$\lim_{h\to 0}\frac{b_{41}}{-h}=\omega\frac{2B_1k_2^2-AB_2k_1}{B_1-B_2},$$

$$\lim_{h\to 0}\frac{b_{42}}{-h}=0,$$

$$\lim_{h\to 0}\frac{b_{43}}{-h}=\omega\frac{Ak_1-2k_2^2}{B_1-B_2},$$

$$\lim_{h\to 0}\frac{b_{44}-1}{-h}=-\frac{d}{dy}\ln\mu(\omega,y).$$

$$(67)\quad \frac{d\delta}{dy}=\omega\frac{2B_1k_2^2-AB_2k_1}{B_1-B_2}\alpha(\omega,y)+\omega\frac{Ak_1-2k_2^2}{B_1-B_2}\gamma(\omega,y)- \\ -\frac{d}{dy}\ln\mu(\omega,y)\delta(\omega,y)$$

Putting together the new equations (64), (65), (66), (67) leads to the following system of ordinary differential equations:

$$
\begin{aligned}
\frac{d\alpha}{dy} &= -\frac{\omega}{k_1^2}\beta(\omega,y)+\omega\delta(\omega,y),\\
\frac{d\beta}{dy} &= \omega k_1^2\frac{B_1k_2^2-B_2}{B_1-B_2}\alpha(\omega,y)+2\frac{d}{dy}\ln k_1(y)\beta(\omega,y)+\omega k_1^2\frac{1-k_1^2}{B_1-B_2}\gamma(\omega,y),\\
\frac{d\gamma}{dy} &= \omega\frac{2B_1k_1-AB_2}{k_1(2-Ak_1)}\beta(\omega,y)-\frac{d}{dy}\ln E(y)\gamma(\omega,y)+\omega\frac{B_2-B_1k_1^2}{2-Ak_1}\delta(\omega,y),\\
\frac{d\delta}{dy} &= \omega\frac{2B_1k_2^2-AB_2k_1}{B_1-B_2}\alpha(\omega,y)+\omega\frac{Ak_1-2k_2^2}{B_1-B_2}\gamma(\omega,y)-\frac{d}{dy}\ln\mu(\omega,y)\delta(\omega,y).
\end{aligned}
$$

11.1. **Lamzyuk-Privarnikov functions in the case of inhomogeneous layer.** Now we have to take the limit in the recurrence relations for Lamzyuk-Privarnikov's functions. It will be a little bit more complicated because these relations are not linear as in the previous case.

First of all we start by introducing continuous notation for Lamzyuk-Privarnikov functions:

$$
\begin{aligned}
A_k(\omega) &\equiv A(\omega,y), \qquad A_{k+1}(\omega)\equiv A(\omega,y+h),\\
B_k(\omega) &\equiv B(\omega,y), \qquad B_{k+1}(\omega)\equiv B(\omega,y+h),\\
C_k(\omega) &\equiv C(\omega,y), \qquad C_{k+1}(\omega)\equiv C(\omega,y+h),\\
D_k(\omega) &\equiv D(\omega,y), \qquad D_{k+1}(\omega)\equiv D(\omega,y+h).
\end{aligned}
$$

Also, earlier we established recurrence relations for these functions:

$$
A(\omega,y+h)=\frac{Y_n(\omega)}{R_n(\omega)},\qquad B(\omega,y+h)=\frac{X_n(\omega)}{R_n(\omega)},
$$

$$
C(\omega,y+h)=\frac{Q_n(\omega)}{R_n(\omega)},\qquad D(\omega,y+h)=\frac{Z_n(\omega)}{R_n(\omega)}.
$$

From both parts of these identities we subtract $A(\omega,y)$, $B(\omega,y)$, $C(\omega,y)$, $D(\omega,y)$ respectively and divide them by h:

$$
\begin{aligned}
\frac{A(\omega,y+h)-A(\omega,y)}{h} &= \frac{Y_n(\omega)-A_n(\omega)R_n(\omega)}{hR_n(\omega)},\\
\frac{B(\omega,y+h)-B(\omega,y)}{h} &= \frac{X_n(\omega)-B_n(\omega)R_n(\omega)}{hR_n(\omega)},\\
\frac{C(\omega,y+h)-C(\omega,y)}{h} &= \frac{Q_n(\omega)-C_n(\omega)R_n(\omega)}{hR_n(\omega)},\\
\frac{D(\omega,y+h)-D(\omega,y)}{h} &= \frac{Z_n(\omega)-D_n(\omega)R_n(\omega)}{hR_n(\omega)}.
\end{aligned}
$$

Taking the limit as $h \to 0$ yields differential equations:

$$\frac{dA}{dy} = \lim_{h\to 0}\frac{Y_n(\omega) - A_n(\omega)R_n(\omega)}{hR_n(\omega)},$$
$$\frac{dB}{dy} = \lim_{h\to 0}\frac{X_n(\omega) - B_n(\omega)R_n(\omega)}{hR_n(\omega)},$$
$$\frac{dC}{dy} = \lim_{h\to 0}\frac{Q_n(\omega) - C_n(\omega)R_n(\omega)}{hR_n(\omega)},$$
$$\frac{dD}{dy} = \lim_{h\to 0}\frac{Z_n(\omega) - D_n(\omega)R_n(\omega)}{hR_n(\omega)}.$$

Let us calculate these limits.

We start by the first equation. In expanded form this limit is:

$$\lim_{h\to 0}\frac{Y_n(\omega,h) - A_n(\omega,y)R_n(\omega,h)}{R_n(\omega,h)} = \lim_{h\to 0}((g_{11}(\omega) + g_{12}(\omega)A_n(\omega)+$$
$$+ g_{13}(\omega)B_n(\omega) + g_{14}(\omega)C_n(\omega) + g_{15}(\omega)D_n(\omega) + g_{16}(\omega)\Delta_n(\omega))-$$
$$- A_n(\omega)(g_{51}(\omega) + g_{52}(\omega)A_n(\omega) + g_{53}(\omega)B_n(\omega) + g_{54}(\omega)C_n(\omega)]+$$
$$+g_{55}(\omega)D_n(\omega)+g_{56}(\omega)\Delta_n(\omega)))/((g_{51}(\omega)+g_{52}(\omega)A_n(\omega)+g_{53}(\omega)B_n(\omega)+$$
$$+ g_{54}C_n(\omega) + g_{55}(\omega)D_n(\omega) + g_{56}(\omega)\Delta_n(\omega))h).$$

The limit of the denominator is:

$$\lim_{h\to 0} g_{51} = k_1(y),$$
$$\lim_{h\to 0} g_{52} = \lim_{h\to 0} g_{53} = \lim_{h\to 0} g_{54} = \lim_{h\to 0} g_{55} = \lim_{h\to 0} g_{56} = 0.$$

Absolute term of equation:

$$\lim_{h\to 0}\frac{g_{11}}{h} = 0$$

Factor of A:

$$\lim_{h\to 0}\frac{g_{12} - g_{51}}{h} = k_1(y)\frac{d}{dy}\ln E(y)$$

Factor of B:

$$\lim_{h\to 0}\frac{g_{13}}{h} = -\frac{\omega}{k_1(y)}$$

Factor of C:

$$\lim_{h\to 0}\frac{g_{14}}{h} = -\frac{k_1k_2\omega(2k_2 - Ak_1)}{B_1 - B_2}$$

Factor of D:

$$\lim_{h\to 0}\frac{g_{15}}{h} = 0$$

Factor of BC:

$$\lim_{h\to 0}\frac{-g_{16}}{h} = 0$$

Factor of AD:

$$\lim_{h\to 0}\frac{g_{16} - g_{55}}{h} = 0$$

Factor of $A(AD-BC)$:

$$\lim_{h\to 0}\frac{g_{56}}{h}=0$$

Factor of A^2:

$$\lim_{h\to 0}\frac{-g_{52}}{h}=0$$

Factor of AB:

$$\lim_{h\to 0}\frac{-g_{53}}{h}=-\omega\frac{AB_2-2B_1k_1}{2-Ak_1}$$

Factor of AC:

$$\lim_{h\to 0}\frac{-g_{54}}{h}=-\omega\frac{k_1(AB_2k_1-2B_1k_2^2)}{B_1-B_2}.$$

So, we obtain the first equation:

(68)

$$\frac{dA}{dy}=\frac{d}{dy}\ln E(y)A(y,\omega)-\frac{\omega}{k_1^2(y)}B(y,\omega)-\omega\frac{k_2(2k_2-Ak_1)}{B_1-B_2}C(y,\omega)-$$
$$-\omega\frac{AB_2-2B_1k_1}{k_1(1-k_1^2)}A(y,\omega)B(y,\omega)-\omega\frac{AB_2k_1-2B_1k_2^2}{B_1-B_2}A(y,\omega)C(y,\omega).$$

Now we perform the calculations to obtain the second equation. We have to compute the limit:

$$\lim_{h\to 0}\frac{X_n(\omega,h)-B_n(\omega,y)R_n(\omega,h)}{R_n(\omega,h)}=\lim_{h\to 0}((g_{21}(\omega)+g_{22}(\omega)A_n(\omega)+$$
$$+g_{23}(\omega)B_n(\omega)+g_{24}(\omega)C_n(\omega)+g_{25}(\omega)D_n(\omega)+g_{26}(\omega)\Delta_n(\omega))-$$
$$-B_n(\omega)(g_{51}(\omega)+g_{52}(\omega)A_n(\omega)+g_{53}(\omega)B_n(\omega)+g_{54}(\omega)C_n(\omega)]+$$
$$+g_{55}(\omega)D_n(\omega)+g_{56}(\omega)\Delta_n(\omega)))/((g_{51}(\omega)+g_{52}(\omega)A_n(\omega)+g_{53}(\omega)B_n(\omega)+$$
$$+g_{54}C_n(\omega)+g_{55}(\omega)D_n(\omega)+g_{56}(\omega)\Delta_n(\omega))h).$$

Absolute term of equation:

$$\lim_{h\to 0}\frac{g_{21}}{h}=-\omega\frac{k_1^3(1-k_2^2)}{B_1-B_2}$$

Factor of A:

$$\lim_{h\to 0}\frac{g_{22}}{h}=-\omega\frac{k_1^3(B_2-B_1k_2^2)}{B_1-B_2}$$

Factor of B:

$$\lim_{h\to 0}\frac{g_{23}-g_{51}}{h}=2\frac{dk_1}{dy}+k_1(y)\frac{dk_1}{dy}\ln E(y)$$

Factor of C:

$$\lim_{h\to 0}\frac{g_{24}}{h}=0$$

Factor of D:

$$\lim_{h\to 0}\frac{g_{25}}{h}=-\omega\frac{k_1(Ak_1-2k_2^2)}{B_1-B_2}$$

Factor of AD:

$$\lim_{h\to 0}\frac{g_{26}}{h}=-\omega\frac{k_1(AB_2k_1-2B_1k_2^2)}{B_1-B_2}$$

Factor of BC:

$$\lim_{h\to 0}\frac{-g_{26}-g_{54}}{h}=0$$

Factor of AB:

$$\lim_{h\to 0}\frac{-g_{52}}{h}=0$$

Factor of B^2:

$$\lim_{h\to 0}\frac{-g_{53}}{h}=\omega\frac{2B_1k_1-AB_2}{2-Ak_1}$$

Factor of BD:

$$\lim_{h\to 0}\frac{-g_{55}}{h}=0$$

Factor of $B(AD-BC)$:

$$\lim_{h\to 0}\frac{-g_{56}}{h}=0.$$

The second equation is

(69)

$$\frac{dB}{dy}=-\omega\frac{k_1^2(1-k_2^2)}{B_1-B_2}-\,-\omega\frac{k_1^2(B_2-B_1k_2^2)}{B_1-B_2}A(y,\omega)+\frac{d}{dy}\ln(k_1^2E)B- \\ -\omega\frac{Ak_1-2k_2^2}{B_1-B_2}D(y,\omega)-\omega\frac{AB_2k_1-2B_1k_2^2}{B_1-B_2}A(y,\omega)D(y,\omega)- \\ -\omega\frac{AB_2-2B_1k_1}{k_1(1-k_1^2)}B^2(y,\omega).$$

We consider the third equation:

$$\lim_{h\to 0}\frac{Q_n(\omega,h)-C_n(\omega,y)R_n(\omega,h)}{R_n(\omega,h)}=\lim_{h\to 0}((g_{31}(\omega)+g_{32}(\omega)A_n(\omega)+ \\ +g_{33}(\omega)B_n(\omega)+g_{34}(\omega)C_n(\omega)+g_{35}(\omega)D_n(\omega)+g_{36}(\omega)\Delta_n(\omega))- \\ -C_n(\omega)(g_{51}(\omega)+g_{52}(\omega)A_n(\omega)+g_{53}(\omega)B_n(\omega)+g_{54}(\omega)C_n(\omega)]+ \\ +g_{55}(\omega)D_n(\omega)+g_{56}(\omega)\Delta_n(\omega)))/((g_{51}(\omega)+g_{52}(\omega)A_n(\omega)+g_{53}(\omega)B_n(\omega)+ \\ +g_{54}C_n(\omega)+g_{55}(\omega)D_n(\omega)+g_{56}(\omega)\Delta_n(\omega))h).$$

Absolute term of equation:

$$\lim_{h\to 0}\frac{g_{31}}{h}=-\omega k_1$$

Factor of A:

$$\lim_{h\to 0}\frac{g_{32}}{h}=-\omega\frac{k_1(B_2-B_1k_1^2)}{2-Ak_1}$$

Factor of B:

$$\lim_{h\to 0}\frac{g_{33}}{h}=0$$

Factor of C:

$$\lim_{h\to 0}\frac{g_{34}-g_{51}}{h}=k_1\frac{d}{dy}\ln\mu(y)$$

Factor of D:

$$\lim_{h\to 0}\frac{g_{35}}{h}=-\frac{\omega}{k_1}$$

Factor of AD:

$$\lim_{h\to 0}\frac{g_{36}}{h}=-\omega\frac{AB_2-2B_1k_1}{2-Ak_1}$$

Factor of BC:

$$\lim_{h\to 0}\frac{-g_{36}-g_{53}}{h}=0$$

Factor of AC:

$$\lim_{h\to 0}\frac{-g_{52}}{h}=0$$

Factor of C^2:

$$\lim_{h\to 0}\frac{-g_{54}}{h}=-\omega\frac{k_1(AB_2k_1-2B_1k_2^2)}{B_1-B_2}$$

Factor of CD:

$$\lim_{h\to 0}\frac{-g_{55}}{h}=0$$

Factor of $C(AD-BC)$:

$$\lim_{h\to 0}\frac{-g_{56}}{h}=0.$$

Here is the third equation:

(70)

$$\frac{dC}{dy}=-\omega-\omega\frac{B_2-B_1k_1^2}{1-k_1^2}A(y,\omega)+\frac{d}{dy}\ln\mu(y)C(y,\omega)-\frac{\omega}{k_1^2}D(y,\omega)-$$
$$-\omega\frac{AB_2-2B_1k_1}{k_1(1-k_1^2)}A(y,\omega)D(y,\omega)-\omega\frac{AB_2k_1-2B_1k_2^2}{B_1-B_2}C^2(y,\omega).$$

And now we consider the last one:

$$\lim_{h\to 0}\frac{Z_n(\omega,h)-D_n(\omega,y)R_n(\omega,h)}{R_n(\omega,h)}=\lim_{h\to 0}((g_{41}(\omega)+g_{42}(\omega)A_n(\omega)+$$
$$+g_{43}(\omega)B_n(\omega)+g_{44}(\omega)C_n(\omega)+g_{45}(\omega)D_n(\omega)+g_{46}(\omega)\Delta_n(\omega))-$$
$$-D_n(\omega)(g_{51}(\omega)+g_{52}(\omega)A_n(\omega)+g_{53}(\omega)B_n(\omega)+g_{54}(\omega)C_n(\omega)]+$$
$$+g_{55}(\omega)D_n(\omega)+g_{56}(\omega)\Delta_n(\omega)))/((g_{51}(\omega)+g_{52}(\omega)A_n(\omega)+g_{53}(\omega)B_n(\omega)+$$
$$+g_{54}C_n(\omega)+g_{55}(\omega)D_n(\omega)+g_{56}(\omega)\Delta_n(\omega))h).$$

Absolute term of equation:

$$\lim_{h\to 0}\frac{g_{41}}{h}=0$$

Factor of A:

$$\lim_{h\to 0}\frac{g_{42}}{h}=0$$

Factor of B:

$$\lim_{h\to 0}\frac{g_{43}}{h} = -\omega\frac{k_1(B_2 - B_1k_1^2)}{1-k_1^2}$$

Factor of C:

$$\lim_{h\to 0}\frac{g_{44}}{h} = -\omega\frac{k_1^3(B_2 - B_1k_2^2)}{B_1 - B_2}$$

Factor of D:

$$\lim_{h\to 0}\frac{g_{45}-g_{51}}{h} = 2\frac{dk_1}{dy} + k_1\frac{d}{dy}\ln\mu(y)$$

Factor of BC:

$$\lim_{h\to 0}\frac{-g_{46}}{h} = 0$$

Factor of AD:

$$\lim_{h\to 0}\frac{g_{46}-g_{52}}{h} = 0$$

Factor of BD:

$$\lim_{h\to 0}\frac{-g_{53}}{h} = \omega\frac{AB_2 - 2B_1k_1}{1-k_1^2}$$

Factor of DC:

$$\lim_{h\to 0}\frac{-g_{54}}{h} = -\omega\frac{k_1(AB_2k_1 - 2k_2^2B_1)}{B_1 - B_2}$$

Factor of D^2:

$$\lim_{h\to 0}\frac{-g_{55}}{h} = 0$$

Factor of $D(AD - BC)$:

$$\lim_{h\to 0}\frac{g_{56}}{h} = 0.$$

The fourth equation reads

$$\begin{aligned}(71)\quad \frac{dD}{dy} = &-\omega\frac{B_2 - B_1k_1^2}{1-k_1^2}B(y,\omega) - \omega\frac{k_1^2(B_2 - B_1k_2^2)}{B_1 - B_2}C(y,\omega) + \\ &+\frac{d}{dy}\ln(k_1\mu)D(y,\omega) - \omega\frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}B(y,\omega)D(y,\omega) + \\ &+\omega\frac{AB_2k_1 - 2k_2^2B_1}{B_1 - B_2}D(y,\omega)C(y,\omega).\end{aligned}$$

After putting together equations (68), (69), (70), (71) we obtain a system of ordinary differential equations of Riccati type:

$$\begin{aligned}
\frac{dA}{dy} &= \frac{d}{dy}\ln E(y)A(y,\omega) - \frac{\omega}{k_1^2(y)}B(y,\omega) - \omega\frac{k_2(2k_2 - Ak_1)}{B_1 - B_2}C(y,\omega) - \\
&\omega\frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}A(y,\omega)B(y,\omega) - \omega\frac{AB_2k_1 - 2B_1k_2^2}{B_1 - B_2}A(y,\omega)C(y,\omega),\\
\frac{dB}{dy} &= -\omega\frac{k_1^2(1-k_2^2)}{B_1 - B_2} - \omega\frac{k_1^2(B_2 - B_1k_2^2)}{B_1 - B_2}A(y,\omega) + \frac{d}{dy}\ln(k_1^2E)B(y,\omega) - \\
&-\omega\frac{Ak_1 - 2k_2^2}{B_1 - B_2}D(y,\omega) - \omega\frac{AB_2k_1 - 2B_1k_2^2}{B_1 - B_2}A(y,\omega)D(y,\omega)\\
&-\omega\frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}B^2(y,\omega),\\
\frac{dC}{dy} &= -\omega - \omega\frac{B_2 - B_1k_1^2}{1-k_1^2}A(y,\omega) + \frac{d}{dy}\ln\mu(y)C(y,\omega) - \frac{\omega}{k_1^2}D(y,\omega) - \\
&-\omega\frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}A(y,\omega)D(y,\omega) - \omega\frac{AB_2k_1 - 2B_1k_2^2}{B_1 - B_2}C^2(y,\omega),\\
\frac{dD}{dy} &= -\omega\frac{B_2 - B_1k_1^2}{1-k_1^2}B(y,\omega) - \omega\frac{k_1^2(B_2 - B_1k_2^2)}{B_1 - B_2}C(y,\omega) + \frac{d}{dy}\ln(k_1\mu)D(y,\omega) - \\
&-\omega\frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}B(y,\omega)D(y,\omega) + \omega\frac{AB_2k_1 - 2k_2^2B_1}{B_1 - B_2}D(y,\omega)C(y,\omega).
\end{aligned}$$

This system of ordinary differential equations plays exactly the same role as recurrence relations for Lamzyuk-Privarnikov functions in discrete case. Similarly the equations of this system depend only on mechanical properties of the layer. Thus, the solution of this system can be used for a number of problems.

The initial conditions for this system of ODE are discussed below.

We would like to mention here an open problem. It is the well known fact that solutions of Riccati type differential equations can achieve the infinity for finite values of argument. Numerical experiences show that this situation can be realized but not always. It would be interesting to know what are the conditions on mechanical properties that do not allow to the solutions to tend to infinity when $y \in [0, h]$. And what is the interpretation of this behavior in context of the theory of elasticity boundary value problems.

11.2. **Asymptotic analysis of the resulting equations.** It is useful to know the asymptotic behavior of the solutions of system (68), (69), (70), (71). More precisely we have to calculate the limit

$$B_\infty := \lim_{\omega\to\infty} B(h,\omega),$$

because it appears in singular integral equation.

In order to be able to calculate it we write asymptotic expansion for $\omega \gg 1$:

$$\begin{aligned}
A(y,\omega) &= A_0(y) + \frac{1}{\omega}A_1(y) + \dots \\
B(y,\omega) &= B_0(y) + \frac{1}{\omega}B_1(y) + \dots \\
C(y,\omega) &= C_0(y) + \frac{1}{\omega}C_1(y) + \dots \\
D(y,\omega) &= D_0(y) + \frac{1}{\omega}D_1(y) + \dots
\end{aligned}$$

and divide the system of equations (68), (69), (70), (71) by ω:

$$\begin{aligned}
\frac{1}{\omega}\frac{dA}{dy} &= \frac{1}{\omega}\frac{d}{dy}\ln E(y)A(y,\omega) - \frac{1}{k_1^2(y)}B(y,\omega) - \frac{k_2(2k_2 - Ak_1)}{B_1 - B_2}C(y,\omega) - \\
&\quad \frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}A(y,\omega)B(y,\omega) - \frac{AB_2k_1 - 2B_1k_2^2}{B_1 - B_2}A(y,\omega)C(y,\omega), \\
\frac{1}{\omega}\frac{dB}{dy} &= -\frac{k_1^2(1-k_2^2)}{B_1 - B_2} - \frac{k_1^2(B_2 - B_1k_2^2)}{B_1 - B_2}A(y,\omega) + \frac{1}{\omega}\frac{d}{dy}\ln(k_1^2E)B(y,\omega) - \\
&\quad -\frac{Ak_1 - 2k_2^2}{B_1 - B_2}D(y,\omega) - \frac{AB_2k_1 - 2B_1k_2^2}{B_1 - B_2}A(y,\omega)D(y,\omega) \\
&\quad -\frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}B^2(y,\omega), \\
\frac{1}{\omega}\frac{dC}{dy} &= -1 - \frac{B_2 - B_1k_1^2}{1-k_1^2}A(y,\omega) + \frac{1}{\omega}\frac{d}{dy}\ln\mu(y)C(y,\omega) - \frac{1}{k_1^2}D(y,\omega) - \\
&\quad -\frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}A(y,\omega)D(y,\omega) - \frac{AB_2k_1 - 2B_1k_2^2}{B_1 - B_2}C^2(y,\omega), \\
\frac{1}{\omega}\frac{dD}{dy} &= -\frac{B_2 - B_1k_1^2}{1-k_1^2}B(y,\omega) - \frac{k_1^2(B_2 - B_1k_2^2)}{B_1 - B_2}C(y,\omega) + \frac{1}{\omega}\frac{d}{dy}\ln(k_1\mu)D(y,\omega) - \\
&\quad -\frac{AB_2 - 2B_1k_1}{k_1(1-k_1^2)}B(y,\omega)D(y,\omega) + \frac{AB_2k_1 - 2k_2^2B_1}{B_1 - B_2}D(y,\omega)C(y,\omega).
\end{aligned}$$

Now we substitute the asymptotic expansions in these equations and we look only at the zeroth order in ω. We easily obtain the following system of algebraic equations:

$$\begin{aligned} -\frac{1}{k_1^2(y)}B_0 - \frac{k_2(2k_2-Ak_1)}{B_1-B_2}C_0 - \frac{AB_2-2B_1k_1}{k_1(1-k_1^2)}A_0B_0 - & \\ \frac{AB_2k_1-2B_1k_2^2}{B_1-B_2}A_0C_0 &= 0, \\ -\frac{k_1^2(1-k_2^2)}{B_1-B_2} - \frac{k_1^2(B_2-B_1k_2^2)}{B_1-B_2}A_0 - \frac{Ak_1-2k_2^2}{B_1-B_2}D_0 - & \\ -\frac{AB_2k_1-2B_1k_2^2}{B_1-B_2}A_0D_0 - \frac{AB_2-2B_1k_1}{k_1(1-k_1^2)}B_0^2 &= 0, \\ -1 - \frac{B_2-B_1k_1^2}{1-k_1^2}A_0 - \frac{1}{k_1^2}D_0 - \frac{AB_2-2B_1k_1}{k_1(1-k_1^2)}A_0D_0 - & \\ \frac{AB_2k_1-2B_1k_2^2}{B_1-B_2}C_0^2 &= 0, \\ -\frac{B_2-B_1k_1^2}{1-k_1^2}B_0 - \frac{k_1^2(B_2-B_1k_2^2)}{B_1-B_2}C_0 - \frac{AB_2-2B_1k_1}{k_1(1-k_1^2)}B_0D_0 + & \\ \frac{AB_2k_1-2k_2^2B_1}{B_1-B_2}D_0C_0 &= 0. \end{aligned}$$

Note that this system does not have a unique solution[11] but only one of its solutions has a physical sense. Unfortunately the author could not solve analytically this system. That is why we used Maple to solve numerically this system. The Maple code listing is cited in Appendix 14.

11.3. **Resume.** In the previous subsections we obtained two systems of ordinary differential equations. The first system

$$\begin{aligned} \frac{d\alpha}{dy} &= -\frac{\omega}{k_1^2}\beta(\omega,y) + \omega\delta(\omega,y), \\ \frac{d\beta}{dy} &= \omega k_1^2\frac{B_1k_2^2-B_2}{B_1-B_2}\alpha(\omega,y) + 2\frac{d}{dy}\ln k_1(y)\beta(\omega,y) + \omega k_1^2\frac{1-k_1^2}{B_1-B_2}\gamma(\omega,y), \\ \frac{d\gamma}{dy} &= \omega\frac{2B_1k_1-AB_2}{k_1(2-Ak_1)}\beta(\omega,y) - \frac{d}{dy}\ln E(y)\gamma(\omega,y) + \omega\frac{B_2-B_1k_1^2}{2-Ak_1}\delta(\omega,y), \\ \frac{d\delta}{dy} &= \omega\frac{2B_1k_2^2-AB_2k_1}{B_1-B_2}\alpha(\omega,y) + \omega\frac{Ak_1-2k_2^2}{B_1-B_2}\gamma(\omega,y) - \frac{d}{dy}\ln\mu(\omega,y)\delta(\omega,y). \end{aligned}$$

gives the vector $\overrightarrow{\alpha}(\omega,y)$. Knowing this vector is equivalent to solving our problem as we established earlier in (63). For this system we have a boundary value problem because at the interface Γ_3 we know two components $\gamma(\omega,h)$, $\delta(\omega,h)$ and at the interface Γ_2 we know $\delta(\omega,0)$ and a linear relation between $\beta(\omega,0)$ and $\gamma(\omega,0)$.

[11] In fact this system has ten solutions in the complex plane.

In order to avoid solving the boundary value problem we used Lamzyuk-Privarnikov functions method. This method provides two conditions on the boundary Γ_3 that we did not have:

$$\alpha(\omega, h) = -A(h,\omega)\gamma(\omega, h) - C(h,\omega)\delta(\omega, h), \tag{72}$$
$$\beta(\omega, h) = -B(h,\omega)\gamma(\omega, h) - D(h,\omega)\delta(\omega, h). \tag{73}$$

To determine these functions we obtained the nonlinear system of ordinary differential equations (68), (69), (70), (71).

We have to establish initial conditions for this system. It is not very difficult. To obtain them we consider the limit when the ice thickness $h \to 0$:

$$\alpha(\omega, 0) = -A(0,\omega)\gamma(\omega, 0) - C(0,\omega)\delta(\omega, 0),$$
$$\beta(\omega, 0) = -B(0,\omega)\gamma(\omega, 0) - D(0,\omega)\delta(\omega, 0).$$

In the limit we do not have any ice plate, thus its displacements (α, β) are identically equal to zero for any applied forces (γ, δ). From this simple consideration we deduce initial conditions for system (68), (69), (70), (71):

$$A(0,\omega) = B(0,\omega) = C(0,\omega) = D(0,\omega) \equiv 0.$$

We are able to formulate the algorithm of our problem solution:

(1) First of all we solve a Cauchy type problem for the system (68), (69), (70), (71) and find the four functions $A(h,\omega)$, $B(h,\omega)$, $C(h,\omega)$, $D(h,\omega)$.
(2) With these functions determined in item (1) we find the two unknown components α and β of the vector $\overrightarrow{\alpha}$ at the boundary Γ_3.
(3) Now we solve a Cauchy type problem to determine the vector $\overrightarrow{\alpha}$.
(4) To obtain displacements and stresses we have to perform inverse Fourier transform.

12. Contact problem

Actually the contact mechanics is a very developed field of the theory of elasticity. The best general references on this topic in the Russian literature are [16, 17]. More mathematical treatment of the contact mechanics boundary value problems is [5].

In the western literature the best monograph on this subject is [6].

We make the following assumptions:

(1) The block is assumed to be rigid.
(2) The bottom of the block is given by the function $y = f(x)$, $x \in [-a, a]$ with the following properties:
    - $f(x) \in C^{(1)}(-a, a)$,
    - $f(x) = f(-x)$, $x \in [-a, a]$ in other words the block is symmetric.

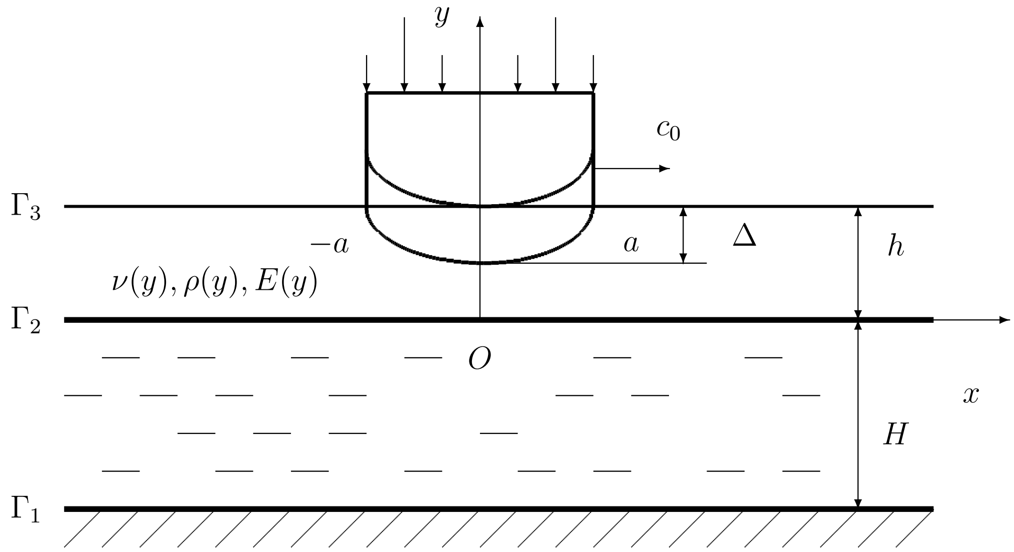

FIGURE 6. Illustration for contact problem formulation.

(3) The block is immerged in the ice plate in such a way that there is full contact at the corner points $(-a, h)$ and (a, h). This assumption gives us an explicit contact zone[12]. This situation is illustrated on Figure 7.
(4) Also in order to simplify the singular integral equation which we will obtain below we suppose that there is no friction between the block and the ice plate.

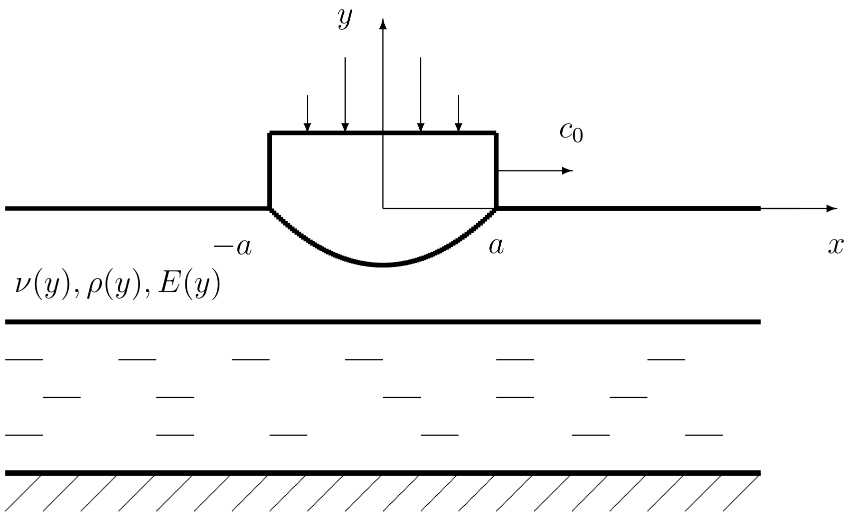

FIGURE 7. Illustration for the immersion of the block in the floating ice plate.

[12]In fact we could consider the more general problem with unknown contact zone but in the frame of this work we decided not to complicate too much. Anyway we should remark that at this point a generalization is possible.

Now we write the mathematical formulation of the contact problem formulation:

$$\sigma_y(x,h) = 0, \qquad |x| > a, \tag{74}$$

$$\tau_{xy}(x,h) = 0, \qquad |x| < \infty, \tag{75}$$

$$v(x,h) = -\Delta + f(x), \qquad |x| < a. \tag{76}$$

Let us explain the meaning of these conditions. The first equation means that there are no forces applied to the upper ice boundary outside of the block in the region $[-a,a]$. The second means the absence of friction on the interface Γ_3. And the last one gives us the vertical ice displacement in the interval $(-a,a)$ on Γ_3. By $\Delta > 0$ we denote the depth at which the block was immerged in the ice (see Figure 6).

12.1. **Singular integral equation derivation.** Earlier we obtained equations (72) and (73) for the continuous case. We are interested here in the second identity (73):

$$\beta(\omega,h) = -B(h,\omega)\gamma(\omega,h) - D(h,\omega)\delta(\omega,h)$$

The vector $\overrightarrow{\alpha}(\omega,y)$ was defined as

$$\begin{aligned}
\hat{u}(\omega,h) &= \alpha(\omega,h),\\
\hat{v}(\omega,h) &= \frac{i}{k_1^2(h)}\beta(\omega,h),\\
\hat{\sigma}_y(\omega,h) &= iE(h)\omega\gamma(\omega,h),\\
\hat{\tau}_{xy}(\omega,h) &= \mu(h)\omega\delta(\omega,h).
\end{aligned}$$

The condition (75) gives us

$$\delta(\omega,h) = \frac{\hat{\tau}_{xy}(\omega,h)}{\mu(h)\omega} = 0.$$

Then equation (73) becomes:

$$\beta(\omega,h) = -B(h,\omega)\gamma(\omega,h). \tag{77}$$

Now we are interested in condition (76). In this condition we do not know the constant Δ. In order to eliminate this constant we take the x-derivative:

$$\frac{dv(x,h)}{dx} = f'(x). \tag{78}$$

A little calculation shows that

$$\left.\widehat{\frac{dv}{dx}}\right|_{y=h} = (-i\omega)\hat{v}(\omega,h) = (-i\omega)\frac{i}{k_1^2(h)}\beta(\omega,h) = \frac{\omega}{k_1^2(h)}\beta(\omega,h).$$

Let us multiply the identity (77) by $\frac{\omega}{k_1^2}$:

$$\frac{\omega}{k_1^2}\beta(\omega,h) = -\frac{\omega}{k_1^2}B(h,\omega)\gamma(\omega,h).$$

Taking the inverse Fourier transform of this equality gives us:

$$\frac{dv(x,h)}{dx} = -\frac{1}{2\pi k_1^2(h)} \int_{-\infty}^{\infty} \omega B(h,\omega)\gamma(\omega,h)e^{-i\omega x}\, d\omega.$$

Using (78) we obtain:

$$(79)\quad -\frac{1}{2\pi k_1^2(h)} \int_{-\infty}^{\infty} \omega B(h,\omega)\gamma(\omega,h)e^{-i\omega x}\, d\omega = f'(x), \qquad x \in (-a,a).$$

In the contact problem the contact pressure is defined by

$$q(x) := -\sigma_y(x,h), \qquad |x| < a.$$

In the beginning of this section we assumed that the block is symmetric. This fact implies that the function $x \mapsto q(x)$ is even. This can be seen from a physical point of view.

By definition $iE(h)\omega\gamma(\omega,h) = \hat{\sigma}_y(\omega,h)$. So, we can make $q(x)$ transparent in our integral equation, since

$$\gamma(\omega,h) = \frac{1}{iE(h)\omega}\hat{\sigma}_y(\omega,h) = \frac{1}{iE(h)\omega}\int_{-\infty}^{\infty} \sigma_y(x,h)e^{i\omega x}\, dx =$$

$$= -\frac{1}{iE(h)\omega}\int_{-a}^{a} q(x)e^{i\omega x}\, dx.$$

Here we used the condition (74). Finally equation (79) becomes

$$-\frac{1}{2\pi k_1^2(h)} \int_{-\infty}^{\infty} \omega B(h,\omega) \left(-\frac{1}{iE(h)\omega} \int_{-a}^{a} q(t)e^{i\omega t}\, dt \right) e^{-i\omega x}\, d\omega = f'(x),$$

or after simplification

(80)

$$\frac{1}{2\pi i k_1^2 E} \int_{-\infty}^{+\infty} B(h,\omega)e^{-i\omega x} \left(\int_{-a}^{a} q(t)e^{i\omega t}\, dt \right) d\omega = f'(x), x \in (-a,a).$$

Lemma 12.1. *The function $\omega \mapsto B(h,\omega)$ is odd.*

Proof. Gap. □

Using Lemma 12.1 and the fact that the function $q(x)$ is even we can transform equation (80) in an equivalent form:

$$\frac{1}{2\pi i k_1^2 E}\int_0^{+\infty} B_h(\omega)\int_{-a}^{a} q(t)e^{i\omega(t-x)}\,dt\,d\omega+$$

$$+\frac{1}{2\pi i k_1^2 E}\int_{-\infty}^{0} B_h(\omega)\int_{-a}^{a} q(t)e^{i\omega(t-x)}dtd\omega = f'(x),$$

where we introduced the notation $B_h(\omega) := B(h,\omega)$, or

$$\frac{1}{2\pi i k_1^2 E}\int_0^{+\infty} B_h(\omega)\int_{-a}^{a} q(t)e^{i\omega(t-x)}\,dt\,d\omega-$$

$$-\frac{1}{2\pi i k_1^2 E}\int_0^{+\infty} B_h(\omega)\int_{-a}^{a} q(t)e^{i\omega(x-t)}\,dt\,d\omega = f'(x),$$

$$\frac{1}{2\pi i k_1^2 E}\int_0^{+\infty} B_h(\omega)\Big(\int_{-a}^{a} q(t)e^{-i\omega(x-t)}\,dt-$$

$$-\int_{-a}^{a} q(t)e^{i\omega(x-t)}\,dt\Big)\,d\omega = f'(x).$$

Since

$$e^{-i\omega(x-t)} - e^{i\omega(x-t)} = -2i\sin(\omega(x-t)),$$

$$\frac{1}{2\pi i k_1^2 E}\int_0^{+\infty} B_h(\omega)\int_{-a}^{a} q(t)\Big(-2i\sin(\omega(x-t))\Big)\,dt\,d\omega = f'(x),$$

$$\frac{1}{\pi k_1^2 E}\int_0^{+\infty} B_h(\omega)\int_{-a}^{a} q(t)\sin(\omega(x-t))\,dt\,d\omega = -f'(x),$$

$$\frac{1}{\pi k_1^2 E}\int_0^{+\infty} B_h(\omega)\Big(\int_0^{a} q(t)\sin(\omega(x-t))\,dt+$$

$$+\int_{-a}^{0} q(t)\sin(\omega(x-t))\,dt\Big)\,d\omega = -f'(x),$$

$$\frac{1}{\pi k_1^2 E}\int\limits_0^{+\infty} B_h(\omega)\Big(\int\limits_0^a q(t)\sin(\omega(x-t))\,dt+$$

$$+\int\limits_0^a q(t)\sin(\omega(x+t))\,dt\Big)\,d\omega=-f'(x),$$

$$\frac{1}{\pi k_1^2 E}\int\limits_0^{+\infty} B_h(\omega)\int\limits_0^a q(t)\Big(\sin(\omega(x-t))\,dt+$$

$$+\sin(\omega(x+t))\Big)\,dt\,d\omega=-f'(x),$$

$$\int\limits_0^{+\infty} B_h(\omega)\int\limits_0^a q(t)\Big(\sin(\omega(x-t))\,dt+ \tag{81}$$

$$+\sin(\omega(x+t))\Big)\,dt\,d\omega=-\pi k_1^2 E f'(x),$$

Now we represent the function $B_h(\omega)$ in the form:

$$B_h(\omega)=B_\infty+(B_h(\omega)-B_\infty),$$

where $B_\infty=\lim_{\omega\to+\infty}B(h,\omega)$. We put this representation in the last integral equation (81):

$$\int\limits_0^{+\infty} B_\infty\int\limits_0^a q(t)\Big(\sin(\omega(x-t))\,dt+\sin(\omega(x+t))\Big)\,dt\,d\omega+$$

$$+\int\limits_0^{+\infty}[B_h(\omega)-B_\infty]\int\limits_0^a q(t)\Big(\sin(\omega(x-t))\,dt+\sin(\omega(x+t))\Big)\,dt\,d\omega=$$

$$=-\pi k_1^2 E f'(x),\qquad x\in(-a,a),$$

$$\int\limits_0^a q(t)\,dt\int\limits_0^{+\infty} B_\infty\sin(\omega(x+t))\,d\omega+\int\limits_0^a q(t)\,dt\int\limits_0^{+\infty} B_\infty\sin(\omega(x-t))\,d\omega+$$

$$+\int\limits_0^a q(t)\,dt\int\limits_0^{+\infty}[B_h(\omega)-B_\infty]\Big(\sin(\omega(x-t))\,dt+\sin(\omega(x+t)\Big)\,d\omega=$$

$$=-\pi k_1^2 E f'(x),\qquad x\in(-a,a),$$

Lemma 12.2.

$$\int\limits_0^{+\infty}\sin(\xi t)d\xi=\frac{1}{t}$$

Proof. Gap. □

At this moment we make use of Lemma 12.2 and obtain:

$$B_\infty \int_0^a \frac{q(t)}{x+t}\,dt + B_\infty \int_0^a \frac{q(t)}{x-t}\,dt +$$
$$+ \int_0^a q(t)\,dt \int_0^{+\infty} [B_h(\omega) - B_\infty] \sin(\omega(x+t))\,d\omega +$$
$$+ \int_0^a q(t)\,dt \int_0^{+\infty} [B_h(\omega) - B_\infty] \sin(\omega(x-t))\,d\omega = -\pi k_1^2 E f'(x),$$

In the integrals with $(x+t)$ we change the variable t to $-t$:

$$B_\infty \int_{-a}^a \frac{q(t)}{x-t}\,dt + \int_{-a}^a q(t)\,dt \int_0^{+\infty} [B_h(\omega) - B_\infty] \sin(\omega(x-t))\,d\omega =$$
$$= -\pi k_1^2 E f'(x), \qquad x \in (-a,a)$$

with obvious notation it is easy to recognize a singular integral equation with Cauchy type singularity:

$$\boxed{\int_{-a}^a \frac{q(t)}{t-x}\,dt + \int_{-a}^a q(t) K(x-t)\,dt = \varphi(x), x \in (-a,a),} \tag{82}$$

where

$$K(z) := \int_0^{+\infty} \Big[1 - \frac{B_h(\omega)}{B_\infty}\Big] \sin(\omega z)\,d\omega \tag{83}$$

and

$$\varphi(x) := \frac{\pi k_1^2(h) E(h)}{B_\infty} f'(x).$$

Note that all integrals in this section should be understood in the sense of Cauchy principal value.

Now we can already start to solve numerically this integral equation, but we would like to achieve one more step. In the next subsection we will transform this equation to an equivalent form.

12.2. **Singular integral equation regularization by Carleman-Vekua method.** In this subsection we apply the well known method of singular integral equation regularization which was proposed by the Georgian mechanist Niko Vekua (1913-1993) in [30]. To our knowledge this problem was posed by the Swedish mathematician Tage Gills Torsten Carleman (1892-1949).

This method allows us to obtain an equivalent Fredholm second kind integral equation. This is very useful because there are a lot of subroutines which were written precisely to solve this type of integral equations. Moreover these equations have very good properties which allow us to solve them without any difficulties.

Let us start. At the end of the last subsection we obtained a singular integral equation with Cauchy type singularity:

$$\int_{-a}^{a} \frac{q(t)}{t-x}\, dt + \int_{-a}^{a} q(t) K(x-t)\, dt = \varphi(x), x \in (-a, a),$$

where

$$K(z) := \int_{0}^{+\infty} \Big[1 - \frac{B_h(\omega)}{B_\infty}\Big] \sin(\omega z)\, d\omega$$

and

$$\varphi(x) := \frac{\pi k_1^2(h) E(h)}{B_\infty} f'(x).$$

Let

$$\Phi(x) := -\int_{-a}^{a} q(t) K(x-t)\, dt + \varphi(x).$$

Then equation (82) becomes

$$\int_{-a}^{a} \frac{q(t)}{t-x}\, dt = \Phi(x), \qquad |x| < a. \tag{84}$$

For now we suppose that the function $\Phi(x)$ is known and we will continue to work as if it were known.

We consider the contour $L = \{z \in \mathbb{C} : |\operatorname{Re} z| < a\}$ in the complex plane. This situation is illustrated on Figure 8

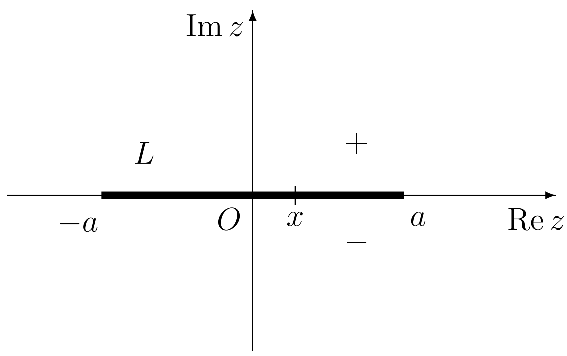

FIGURE 8. Illustration for contour in complex plane.

We consider the complex function

$$F(z) := \frac{1}{2\pi i} \int\limits_L \frac{q(t)}{t - z}\, dt \tag{85}$$

and two functions linked with $F(z)$ by following definitions:

$$F^+(x) := \lim_{z \to x+0} F(z), \qquad x \in L,$$

$$F^-(x) := \lim_{z \to x-0} F(z), \qquad x \in L.$$

These definitions are illustrated on Figure 9.

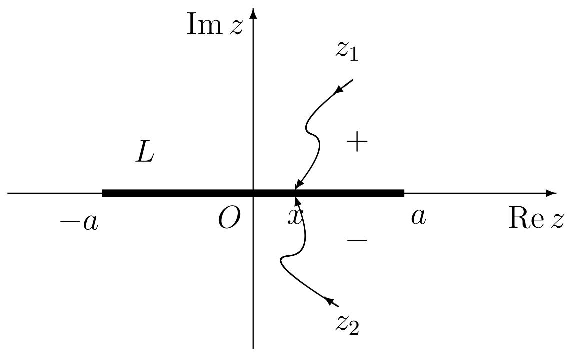

FIGURE 9. Illustration for the definition of $F^+(x)$, $F^-(x)$.

In the russian literature on complex analysis (for example [23, 11]) you can find two formulae that were proved by russian mathematician Yulian Vasilievich Sokhotski (1842-1927):

$$F^+(x) = \frac{1}{2} q(x) + \frac{1}{2\pi i} \int\limits_L \frac{q(t)}{t - x}\, dt, \tag{86}$$

$$F^-(x) = -\frac{1}{2} q(x) + \frac{1}{2\pi i} \int\limits_L \frac{q(t)}{t - x}\, dt. \tag{87}$$

We will rewrite these formulae in equivalent form:

$$F^+(x) + F^-(x) = \frac{1}{\pi i} \int\limits_L \frac{q(t)}{t - x}\, dt, \tag{88}$$

$$F^+(x) - F^-(x) = q(x). \tag{89}$$

Recall that from (84)

$$\int\limits_L \frac{q(t)}{t - x}\, dt = \Phi(x).$$

Substituting it in (88) gives us another formulation of our problem. More exactly, we have Riemann problem for analytical function $F(z)$:

$$\boxed{F^+(x) + F^-(x) = \frac{1}{\pi i}\Phi(x), \quad x \in (-a, a)} \tag{90}$$

Here we would like to make a remark about the Riemann problem. In general case a Riemann problem is formulated like this. With given analytic functions $g(z)$ and $G(z)$ we have to find an analytic function $F(z)$ in $\mathbb{C}\backslash L$ which satisfies this relation on L:

$$F^+(z) + g(z)F^-(z) = G(z), \quad z \in L.$$

In other words Riemann problem consists in reconstruction of analytic function from its discontinuity on a contour.

If $G(z) \equiv 0$ we say that Riemann problem is homogeneous.

So, we transformed our integral equation to the Riemann problem (90). Now we want to solve this problem. For this we need the following technical lemma:

Lemma 12.3. *If a function $F(z)$ is defined by (85) then we have the following asymptotic behavior as $z \to \infty$:*

$$F(z) = -\frac{Q}{2\pi i z} + O\Big(\frac{1}{z^3}\Big),$$

where Q is total load applied on the block:

$$Q := \int_{-a}^{a} q(t)\, dt.$$

Proof.

$$F(z) = \frac{1}{2\pi i}\int_L \frac{q(t)}{t-z}\, dt = -\frac{1}{2\pi i z}\int_{-a}^{a} \frac{q(t)}{1-\frac{t}{z}}\, dt =$$

$$= -\frac{1}{2\pi i z}\int_{-a}^{a} q(t)\Big(1-\frac{t}{z}\Big)^{-1} dt =$$

$$= -\frac{1}{2\pi i z}\int_{-a}^{a} q(t)\Big[1+\frac{t}{z}-\frac{t^2}{z^2}+O\Big(\frac{1}{z^3}\Big)\Big]\, dt =$$

$$= -\frac{1}{2\pi i z}\int_{-a}^{a} q(t)\, dt + O\Big(\frac{1}{z^3}\Big) = -\frac{Q}{2\pi i z} + O\Big(\frac{1}{z^3}\Big)$$

□

From this Lemma we immediately obtain a corollary:

$$F(\infty) = \lim_{z\to\infty} F(z) = 0.$$

Later we will use it.

Now we will consider an auxiliary Riemann problem. It will become clear later why we do this.

We want to find an analytical function $X(z)$ which can have singular points only on the contour L and satisfies the relation:

$$X^+(x) + X^-(x) = 0, \quad x \in (-a, a) \tag{91}$$

or in equivalent form

$$\frac{X^+(x)}{X^-(x)} = -1, \quad x \in (-a, a). \tag{92}$$

We consider the function

$$X(z) = \frac{1}{\sqrt{(z+a)(z-a)}}$$

and we prove that this function is the solution of the auxiliary Riemann problem (91).

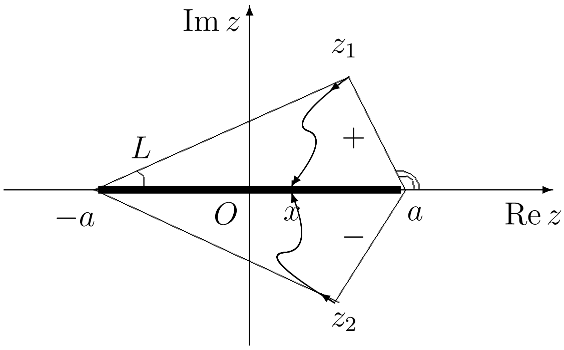

Figure 10. Auxiliary Riemann problem solution.

We compute explicitly the limits $X^+(x)$ and $X^-(x)$:

$$X^+(x) = \lim_{z \to x+0} = \frac{1}{\sqrt{|x+a|e^{i\frac{0}{2}}}\sqrt{|x-a|e^{i\frac{\pi}{2}}}} = \frac{1}{i\sqrt{a^2-x^2}},$$

$$X^-(x) = \lim_{z \to x-0} = \frac{1}{\sqrt{|x+a|e^{i\frac{0}{2}}}\sqrt{|x-a|e^{-i\frac{\pi}{2}}}} = -\frac{1}{i\sqrt{a^2-x^2}}$$

and if we divide $X^+(x)$ by $X^-(x)$:

$$\frac{X^+(x)}{X^-(x)} = \frac{\frac{1}{i\sqrt{a^2-x^2}}}{-\frac{1}{i\sqrt{a^2-x^2}}} = -1.$$

Thus, we proved that the function $X(z)$ is a solution of the problem (91).

Let us return to the main Riemann problem. Its definition is

$$F^+(x) + F^-(x) = \frac{1}{\pi i}\Phi(x).$$

We perform some simple transformations and use the second definition (92) of the function $X(z)$:

$$F^+(x) - (-1)F^-(x) = \frac{1}{\pi i}\Phi(x),$$

$$F^+(x) - \frac{X^+(x)}{X^-(x)}F^-(x) = \frac{1}{\pi i}\Phi(x), \quad x \in (-a, a)$$

$$\frac{F^+(x)}{X^+(x)} - \frac{F^-(x)}{X^-(x)} = \frac{1}{\pi i}\frac{\Phi(x)}{X^+(x)}, \quad x \in (-a, a).$$

Introducing a new complex function

$$\Psi(z) = \frac{F(z)}{X(z)}$$

we have

$$\Psi^+(x) - \Psi^-(x) = \frac{1}{\pi i}\frac{\Phi(x)}{X^+(x)}. \tag{93}$$

In other words we obtain a third Riemann problem for the function $\Psi(z)$. Fortunately we know its solution because it is given by Sokhotski formula (89):

$$F^+(x) - F^-(x) = q(x).$$

Now we are able to write the solution of the third Riemann problem (93):

$$\Psi(z) = \frac{1}{2\pi i}\int\limits_{-a}^{a} \frac{1}{\pi i}\frac{\Phi(t)}{X^+(t)(t-z)}\, dt + P(z),$$

where $P(z) \in \mathbb{C}[z]$ is an arbitrary polynomial.

If we remember the definition of function $\Psi(z) := \frac{F(z)}{X(z)}$ we can rewrite our solution:

$$F(z) = \frac{X(z)}{2\pi i \cdot i\pi}\int\limits_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-z)}\, dt + P(z)X(z).$$

Earlier we obtained a corollary that

$$F(\infty) = 0.$$

This condition implies that polynomial $P(z)$ is a polynomial of the first degree, i.e. $P(z) = c_0$:

$$F(z) = \frac{X(z)}{2\pi i \cdot i\pi}\int\limits_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-z)}\, dt + c_0X(z).$$

To compute this unknown constant c_0 we will write one more time the asymptotic behavior of the function $F(z)$ for $z \gg 1$ from the last formula[13]:

$$F(z) = c_0(z^2 - a^2)^{-\frac{1}{2}} + O\Big(\frac{1}{z^2}\Big) =$$
$$= \frac{c_0}{z}\left(1 - \frac{a^2}{z^2}\right)^{-\frac{1}{2}} + O\Big(\frac{1}{z^2}\Big) = \frac{c_0}{z} + O\Big(\frac{1}{z^2}\Big).$$

In Lemma 12.3 we established that

$$F(z) = -\frac{Q}{2\pi i z} + O\Big(\frac{1}{z^3}\Big)$$

Comparing the last two formulae one can conclude that

$$c_0 = -\frac{Q}{2\pi i}$$

So, we have

$$F(z) = \frac{X(z)}{2\pi i \cdot i\pi}\int_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-z)}\, dt - \frac{Q}{2\pi i}X(z).$$

Now we consider one more complex function

$$\Omega(z) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-z)}\, dt$$

We apply Sokhotski formulae to this function:

$$\Omega^+(x) = \frac{1}{2}\frac{\Phi(x)}{X^+(x)} + \frac{1}{2\pi i}\int_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-x)}\, dt, \quad x \in (-a, a),$$

$$\Omega^-(x) = -\frac{1}{2}\frac{\Phi(x)}{X^+(x)} + \frac{1}{2\pi i}\int_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-x)}\, dt, \quad x \in (-a, a).$$

At this moment we are able to find the limits $F^+(x)$ and $F^-(x)$:

$$F^+(x) = \frac{X^+(x)}{i\pi}\left[\frac{1}{2}\frac{\Phi(x)}{X^+(x)} + \frac{1}{2\pi i}\int_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-x)}\, dt\right] - \frac{Q}{2\pi i}X^+(x),$$

$$F^-(x) = \frac{X^-(x)}{i\pi}\left[-\frac{1}{2}\frac{\Phi(x)}{X^+(x)} + \frac{1}{2\pi i}\int_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-x)}\, dt\right] - \frac{Q}{2\pi i}X^-(x).$$

---

[13]Since $X(z) = O\left(\frac{1}{z}\right)$ and $\frac{1}{2\pi i \cdot i\pi}\int_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-z)}\, dt = O\left(\frac{1}{z}\right)$ we have that $\frac{X(z)}{2\pi i \cdot i\pi}\int_{-a}^{a} \frac{\Phi(t)}{X^+(t)(t-z)}\, dt = O\left(\frac{1}{z^2}\right)$

Recall that the function $X(z)$ is a solution of the Riemann problem (91). By definition

$$X^-(x) = -X^+(x)$$

and the last formula can be rewritten as

$$F^+(x) = \frac{X^+(x)}{i\pi}\left[\frac{1}{2}\frac{\Phi(x)}{X^+(x)} + \frac{1}{2\pi i}\int_{-a}^{a}\frac{\Phi(t)}{X^+(t)(t-x)}\,dt\right] - \frac{Q}{2\pi i}X^+(x),$$

$$F^-(x) = -\frac{X^+(x)}{i\pi}\left[-\frac{1}{2}\frac{\Phi(x)}{X^+(x)} + \frac{1}{2\pi i}\int_{-a}^{a}\frac{\Phi(t)}{X^+(t)(t-x)}\,dt\right] + \frac{Q}{2\pi i}X^+(x)$$

According to Sokhotski formula (89) we have

$$F^+(x) - F^-(x) = q(x).$$

If we subtract expression of $F^-(x)$ from $F^+(x)$ we obtain:

$$q(x) = \frac{X^+(x)}{(\pi i)^2}\int_{-a}^{a}\frac{\Phi(t)}{X^+(t)(t-x)} - \frac{Q}{\pi i}X^+(x).$$

Earlier we performed a simple computation and established that

$$X^+(x) = \frac{1}{i\sqrt{a^2-x^2}}.$$

We put this expression in the integral equation:

$$q(x) = -\frac{1}{\pi^2\sqrt{a^2-x^2}}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}\Phi(t)}{t-x}\,dt + \frac{Q}{\pi\sqrt{a^2-x^2}}. \tag{94}$$

The last step is to recall the definition of the function $\Phi(t)$ that we considered to be known:

$$\Phi(t) = -\int_{-a}^{a} q(s)K(t-s)\,ds + \pi k_1^2 E f'(t).$$

We put this expression in equation (94):

$$q(x) = -\frac{1}{\pi^2\sqrt{a^2-x^2}}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-x}\Big[-\int_{-a}^{a} q(s)K(t-s)\,ds + \\ + \pi k_1^2 E f'(t)\Big] + \frac{Q}{\pi\sqrt{a^2-x^2}},\quad x\in(-a,a).$$

It is now very easy to obtain an equivalent Fredholm second kind integral equation:

$$q(x) - \frac{1}{\pi^2\sqrt{a^2-x^2}} \int_{-a}^{a} \overline{K}(x,t) q(t)\, dt = \frac{Q}{\pi\sqrt{a^2-x^2}} -$$

$$- \frac{k_1^2 E}{\pi\sqrt{a^2-x^2}} \int_{-a}^{a} \frac{\sqrt{a^2-t^2}}{t-x} f'(t)\, dt,$$

where

$$\overline{K}(x,t) = \int_{-a}^{a} \frac{\sqrt{a^2-s^2}K(s-t)}{s-x}\, ds$$

So, at the end of this section we obtained an equivalent Fredholm second kind integral equation which can be solved by standard subroutines without any problem.

But in the present work we will take another way. Personally, we find it more interesting to solve numerically the singular integral equation. The next subsection is devoted to this topic.

12.3. **Numerical algorithm for solving the singular integral equation with Cauchy type singularity.** A. A. Korneychuk obtained a numerical integration formula for singular integrals with Cauchy type kernel:

$$\int_{-1}^{1} \frac{\omega(t)u(t)\, dt}{t-x_r} = \sum_{m=1}^{M} \frac{a_m u(t_m)}{t_m - x_r}. \tag{95}$$

In this formula $u(t)$ is a regular function, $\omega(t)$ is an integrable function which contains the singularity. The integration points $t = t_m$, $(m = 1, 2, \dots, M)$ are the roots of the orthogonal polynomial $P_M(t)$ of degree M with weight function $\omega(t)$, $t \in [-1, 1]$. The points $x = x_r$, $r = 1, 2, \dots, R$ are the roots of the function

$$Q_M(x) = -\frac{1}{2} \int_{-1}^{1} \frac{\omega(t) P_M(t)\, dt}{t-x}. \tag{96}$$

The coefficients a_m, $(m = 1, 2, \dots, M)$ of the numerical integration formula (95) are determined by the following formula

$$a_m = -2\frac{Q_M(t_m)}{P'_M(t_m)}.$$

Let us notice that the formula (95) is necessarily exact if $u(t)$ is a polynomial of degree $2M$ or less.

One can notice that the coefficients a_m and the points t_m in the formula (95) coincide with Gauss numerical integration rule for the same weight $\omega(t)$:

$$\int_{-1}^{1} \omega(t)u(t)\,dt = \sum_{m=1}^{M} a_m u(t_m).$$

Therefore, the formula (95) can be considered as Gauss numerical integration formula for singular integral that is true for discret system of points x_r such that $Q_M(x_r) = 0$.

Let us consider the singular integral equation

$$\int_{-1}^{1} \left[\frac{1}{t-x} + K(t,x)\right] g(t)\,dt = \pi p(x), \qquad |x| < 1 \tag{97}$$

where the functions $K(t,x)$ and $p(x)$ are continuous at $[-1,1]^2$ and $[-1,1]$ respectively.

The singular integral equations of this type arise very often in the problems of elasticity, hydro and aerodynamics.

We will make a structural assumption about the solution of the integral equation (97)

$$g(t) = \omega(t)u(t) \tag{98}$$

where $u(t)$ is a regular unknown function and $\omega(t)$ is a known, non negative weight function.

Let us put the representation (98) into the integral equation (97)

$$\int_{-1}^{1} \frac{\omega(t)u(t)\,dt}{t-x} + \int_{-1}^{1} \omega(t)u(t)K(t,x)\,dt = \pi p(x), \qquad |x| < 1. \tag{99}$$

Since the functions $u(t)$, $K(x,t)$ are regular, we can approximate the integrals in (99) at the points x_r, $r = 1, \dots, R$ and obtain a system of R linear algebraic equations:

$$\sum_{m=1}^{M} a_m u(t_m) \left[\frac{1}{t_m - x_r} + k(t_m, x_r)\right] = \pi p(x_r), \qquad (r = 1, 2, \dots, R). \tag{100}$$

There are two possibilities:

(1) $R \geq M$,
(2) $R < M$.

In the first case one should choose exactly M equations from (100) and, after solving them, find unknown function at the points $t = t_m$.

If $R < M$ then we should use some complementary conditions in order to obtain a complete system of equations.

Now let us consider a particular case which is particularly interesting in the context of this work. Moreover this is a seldom case when we can find the coefficients a_m and the points t_m, x_r analytically.

In our problem the solution has the form

$$g(x) = \frac{u(x)}{\sqrt{1-x^2}}.$$

It can be unclear why we have chosen this particular form of the solution. In fact, this can be established by solving a simple problem about the contact between a rigid block and elastic half-space. The solution of this problem can be obtained in analytic form.

In this case $\omega(x) = (1-x^2)^{-\frac{1}{2}}$ and the system of orthogonal polynomial is the system of Tchebychev polynomials of the first kind

$$T_n(x) = \cos(n \arccos x). \tag{101}$$

As we established above, the roots of the polynomials (101) are the integration points:

$$t_m = \cos\frac{2m-1}{2M}\pi$$

and the coefficients

$$a_m = \frac{\pi}{M}, \qquad m = 1, 2, \ldots, M.$$

Using the formula

$$\frac{1}{\pi}\int\limits_{-1}^{1} \frac{T_M(t)\,dt}{\sqrt{1-t^2}(t-x)} = \begin{cases} 0, & M = 0, \\ u_{M-1}(x), & M > 0. \end{cases} \qquad |x| < 1$$

and the formula (96) one obtains

$$Q_M(x) = -2\pi U_{M-1}(x) = -2\pi\frac{\sin(M\arccos x)}{\sqrt{1-x^2}} \tag{102}$$

where we introduced a notation

$$U_{M-1}(x) = \frac{\sin(M\arccos x)}{\sqrt{1-x^2}}.$$

One can recognize that $U_{M-1}(x)$ is the Tchebychev polynomial of the second kind.

As the points x_r are the roots of $Q_M(x)$ we can easily find from (102)

$$x_r = \cos\frac{\pi r}{M}, \qquad r = 1, \ldots, M-1.$$

In this case $R = M - 1 < M$, therefore we have M unknowns $u(t_m)$ and $M-1$ equations (100). But fortunately we have one more condition to obtain a complete system of equations

$$\int\limits_{-1}^{1} \frac{u(t)\,dt}{\sqrt{1-t^2}} = Q \tag{103}$$

where Q is the known constant with physical meaning. It represents the total load applied on the moving block. It is easy to obtain a finite-dimensional analog of the condition (103) applying Gauss numerical integration rule

$$\int_{-1}^{1} \frac{u(t)\,dt}{\sqrt{1-t^2}} = \sum_{m=1}^{M} a_m u(t_m) = \sum_{m=1}^{M} \frac{\pi}{M} u(t_m) = Q \tag{104}$$

So, putting together equations (100) and (104) gives us a system of linear algebraic equations

$$\begin{aligned} \sum_{m=1}^{M} a_m u(t_m) \left[\frac{1}{t_m - x_r} + K(t_m, x_r) \right] &= \pi p(x_r), \qquad (r = 1, 2, \ldots, M-1) \\ \sum_{m=1}^{M} \frac{\pi}{M} u(t_m) &= Q. \end{aligned}$$

12.4. **Dimensionless form of the integral equation.** In order to perform numerical computations we need to transform the integral equation (82) into dimensionless form. To do this, we change the variables:

$$x = \overline{x}a, \qquad t = \overline{t}a, \qquad \omega = \overline{\omega}h$$

where $2a$ is the block length and h is the ice thickness. Putting new variables in the equation (82) gives

$$\int_{-1}^{1} \frac{q(\overline{t}a)\,d\overline{t}}{\overline{t} - \overline{x}} + a \int_{-1}^{1} q(\overline{t}a) K(a(\overline{x} - \overline{t}))\,d\overline{t} = \frac{\pi k_1^2(h) E(h)}{B_\infty} f'(x) \,|_{x=\overline{x}a}$$

We are looking for solution of the form

$$q(x) = \frac{u(x)}{\sqrt{a^2 - x^2}}. \tag{105}$$

We know that the function $q(x)$ satisfies the condition

$$\int_{-a}^{a} q(x)\,dx = \int_{-a}^{a} \frac{u(x)}{\sqrt{a^2 - x^2}}\,dx = Q,$$

or, in new variables

$$\int_{-1}^{1} \frac{u(\overline{x}a)}{\sqrt{1 - \overline{x}^2}}\,d\overline{x} = Q. \tag{106}$$

If we introduce a new function

$$\overline{u}(\overline{x}) = Q u(\overline{x})$$

we will have the condition (106) in the dimensionless form

$$\int_{-1}^{1} \frac{\overline{u}(\overline{x}a)}{\sqrt{1-\overline{x}^2}}\, d\overline{x} = 1. \tag{107}$$

Let us transform slightly the kernel $K(x,t)$ of the integral equation

$$K(a(\overline{x}-\overline{t})) = \int_{0}^{+\infty} \left(1 - \frac{B_h(\omega)}{B_\infty}\right) \sin(\omega a(\overline{x}-\overline{t}))\, d\omega =$$

$$= \frac{1}{h}\int_{0}^{+\infty} \left(1 - \frac{B(h,\frac{\omega}{h})}{B_\infty}\right) \sin(\frac{a}{h}\overline{\omega}(\overline{x}-\overline{t}))\, d\overline{\omega}.$$

In order to simplify the notation we omit below all the dashes.

Finally we can write the dimensionless singular integral equation:

$$\int_{-1}^{1} \frac{\frac{u(t)}{\sqrt{1-t^2}}\, dt}{t-x} + \kappa \int_{-1}^{1} \frac{u(t)}{\sqrt{1-t^2}} K(x-t)\, dt = \frac{a\pi k_1^2(h)E(h)}{QB_\infty}\, f'(x)|_{x=a\overline{x}}$$

where

$$\kappa := \frac{a}{h}.$$

Together with a complementary condition (107) we have a problem that is ready to be solved by the numerical algorithm described in the previous section.

13. Appendix

13.1. Lamé's equations solution for an homogeneous ice layer.

```
% This code determines elastic deformations of an
% homogeneous ice-plate of finite thickness h lying on a
% compressible water. Water depth is H=constant. The load
% is assumed to be a finite block that moves along
% x-axis with velocity c0.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author: Denys Dutykh

tic

sprintf ('Initialisation....')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% CONSTANT DECLARATION
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
N = 65536;       % number of discretisation subintervals
delta = 0.001;   % frequency discretisation step
```

```
a = 1.0;        % a half length of the moving block
h = 2.0*a;      % depth of ice plate
H = 5.0*a;      % water depth
nu = 0.33;      % Poisson ratio
E = 9.5e9;      % Young module
g = 9.80665;    % The standard gravitational acceleration
c0 = 15.0;      % block velocity
gamma = 1500.0; % sound velocity in the water
rhog = 926.0;   % ice density
rhow = 1027.0;  % sea water density
P0 = 17500.0;   % block load
mu = E/(2*(1+nu));                  % Lame's constant
%%%%%%%%%%%3%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% useful constants
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
chi = sqrt(1-(c0/gamma)^2);
k1 = sqrt(1-c0^2*rhog/mu);
k2 = sqrt(1-c0^2*rhog*(1-nu-2*nu^2)/(E*(1-nu)));
P = k2*k1^2;
B1 = 1;
B2 = ((1-nu)*k2^2-nu)/(1-2*nu);
A1 = k1 + 1/k1; A2 = 2*k2;

sprintf ('Linear system solving....')
c = zeros (4,2*N-1);  % uknown constants
% Fourier transforms of displacements
uf = zeros (4,2*N-1);
% discretisation in Fourier domain
range = (-N+1)*delta:delta:(N-1)*delta;
A = zeros(4); b = zeros(4,1); j = 0;
for omega = range, j = j+1;
    if (abs(omega) < 3)
        if (abs(omega) < 40*eps)
        % right member of linear system
            b(1) = -i*(P0/E)*(a/omega);
        else
        % right member of linear system
            b(1) = -i*P0/E*sin(omega*a)/omega^2;
        end
        % calculation of the matrix
        A(1,1) = omega*B1*cosh(k1*omega*h);
        A(1,2) = omega*B1*sinh(k1*omega*h);
        A(1,3) = omega*B2*sinh(k2*omega*h);
        A(1,4) = omega*B2*cosh(k2*omega*h);
```

```
    A(2,1) = A1*sinh(k1*omega*h);
    A(2,2) = A1*cosh(k1*omega*h);
    A(2,3) = A2*cosh(k2*omega*h);
    A(2,4) = A2*sinh(k2*omega*h);

    A(3,2) = A1;
    A(3,3) = A2;

    s = (rhog*omega*c0^2/chi*cosh(chi*omega*H)+...
        rhow*g*sinh(chi*omega*H));

    A(4,1) = omega*B1*sinh(chi*omega*H);
    A(4,2) = -s/(k1*E);
    A(4,3) = -P*s/(k1^2*E);
    A(4,4) = omega*B2*sinh(chi*omega*H);
else
    % right member of linear system
    b(1) = -i*P0*sin(omega*a)/...
    (E*omega^3*cosh(k1*omega*h));

    % calculation of the matrix
    A(1,1) = B1;
    A(1,2) = B1*tanh(k1*omega*h);
    A(1,3) = B2*(sinh(k2*omega*h)/cosh(k1*omega*h));
    A(1,4) = B2*(cosh(k2*omega*h)/cosh(k1*omega*h));

    A(2,1) = A1*tanh(k1*omega*h);
    A(2,2) = A1;
    A(2,3) = A2*(cosh(k2*omega*h)/cosh(k1*omega*h));
    A(2,4) = A2*(sinh(k2*omega*h)/cosh(k1*omega*h));

    A(3,2) = A1;
    A(3,3) = A2;

    s = (rhog*omega*c0^2/chi*coth(chi*omega*H)+...
    rhow*g);

    A(4,1) = B1;
    A(4,2) = -s/(omega*k1*E);
    A(4,3) = -P*s/(omega*k1^2*E);
    A(4,4) = B2;
end

% we find unknown constants
c(:,j) = inv(A)*b;
```

```
    for k=1:4
        if (isnan(c(k,j))|abs(c(k,j))>1e-3) % 8e-1
            c(k,j) = c(k,j-1);
        end
    end

    I = diag([1,i/k1^2,1,i/k1^2]);
    aux = [1,0,0,1; 0,k1,P,0; cosh(k1*omega*h),...
      sinh(k1*omega*h),sinh(k2*omega*h),cosh(k2*omega*h);
      k1*sinh(k1*omega*h),k1*cosh(k1*omega*h),...
      P*cosh(k2*omega*h),P*sinh(k2*omega*h)];

    % here we find displacements Fourier transform
    uf(:,j) = I*aux*c(:,j);
end

Delta = delta/(2*pi);
deltax = 1/(N*Delta); % discretisation step
% x-variable discretisation
rangex = -1/(2*Delta):deltax:1/(2*Delta);

sprintf('Fast Fourier Transform...')

% to understand this command type: doc fftw
fftw ('planner', 'patient');

un1 = uf(:,N-(0:N-1)); un2 = uf(:,N:2*N-1);

% integral calculation by DFT
u0=(Delta*(N*ifft(un2')-fft(un1')))';

u = zeros(4,N+1);

% we form the solution
u(:,1:N/2) = real (u0(:,(N/2+1):N));
u(:,(N/2+1):(N+1)) = real(u0(:,1:(N/2+1)));

% interface between water and ice
dzeta0 = [rangex+u(1,:); u(2,:)];
% interface between air and ice
dzetah = [rangex+u(3,:); u(4,:)];

toc
```

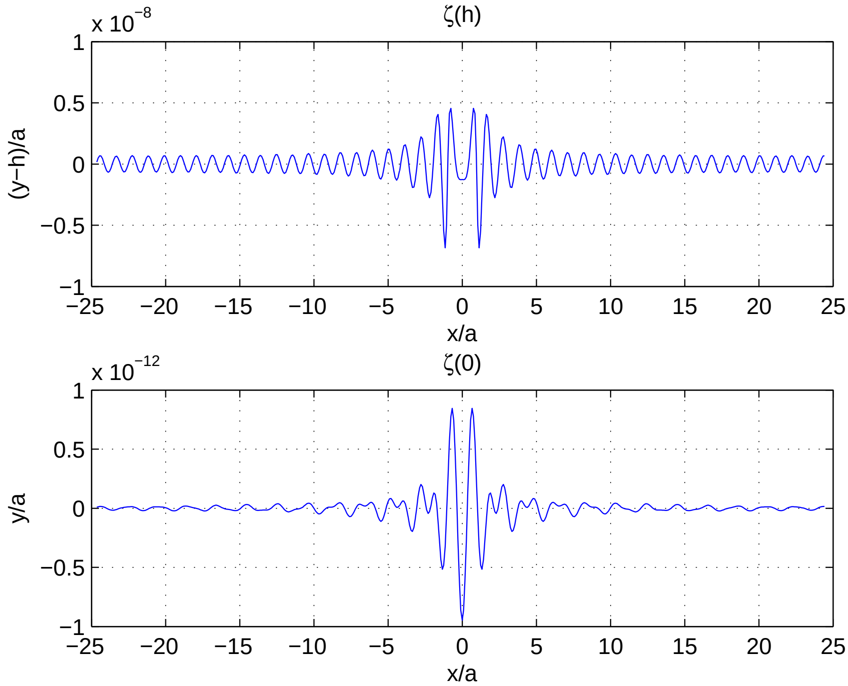

FIGURE 11. Ice deflection for $c_0 = 0.1m/s$

```
sprintf('Plotting....')

subplot (2,1,1)
plot([dzetah(1,(127*N/256):(129*N/256))],
[dzetah(2,(127*N/256):(129*N/256))]),
grid on, title '\zeta(h)', xlabel 'x/a', ylabel '(y-h)/a';
subplot (2,1,2)
plot ([dzeta0(1,(127*N/256):(129*N/256))],
[dzeta0(2,(127*N/256):(129*N/256))]),
grid on, title '\zeta(0)', xlabel 'x/a', ylabel 'y/a';

sprintf('Done.')
```

13.2. **Numerical computations for one homogeneous layer.** In this Appendix you can find the results of numerical computations that were performed with MatLab-code from Appendix 13.1. We tried to show the different types of the ice deflection for different values of the moving block velocity. The results are given on the Figures 11 - 18.

These graphics represent the function $\zeta(x, h)$ and $\zeta(x, 0)$ of ice-air and ice-water interfaces respectively.

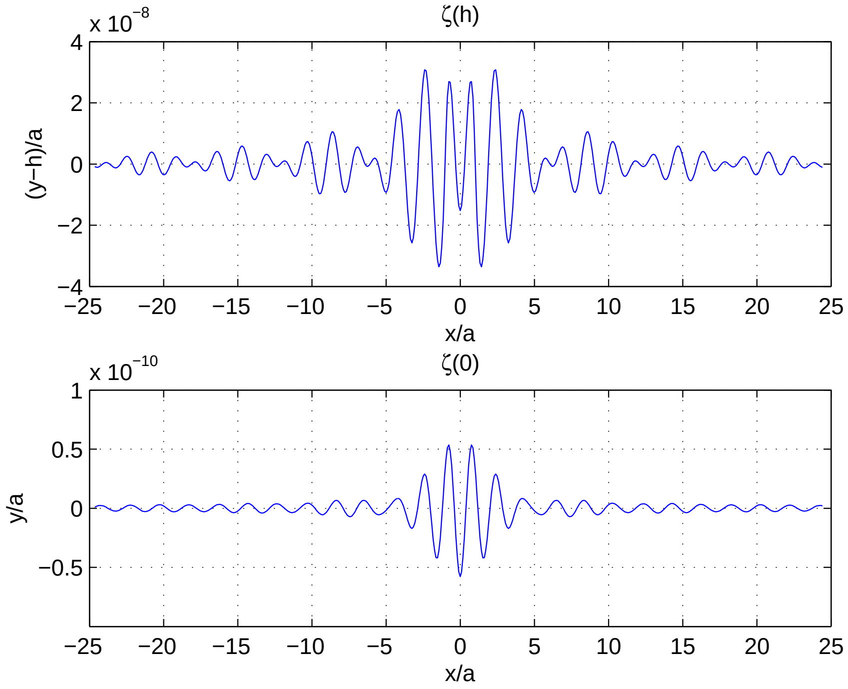

FIGURE 12. Ice deflection for $c_0 = 1.0m/s$

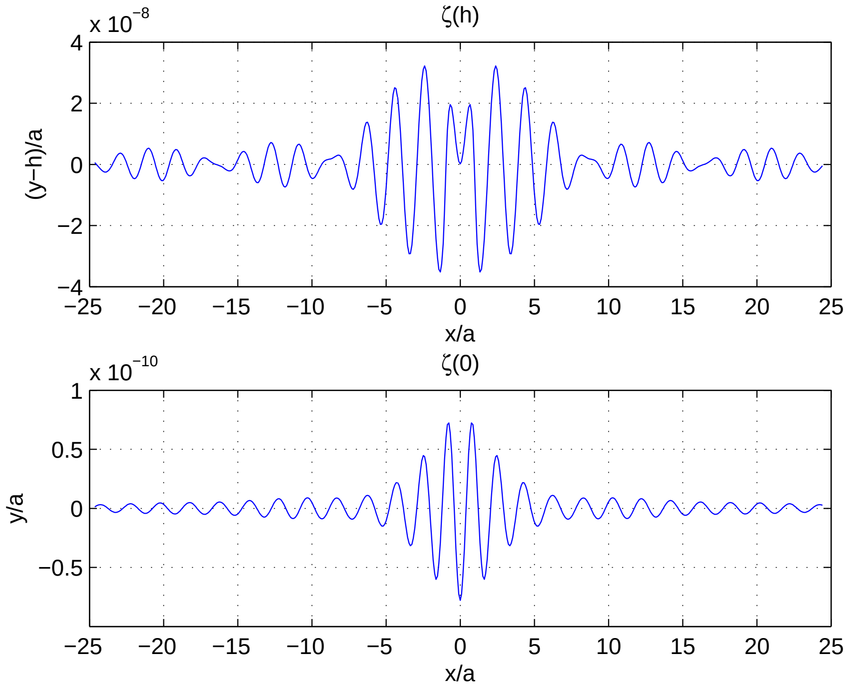

FIGURE 13. Ice deflection for $c_0 = 1.3m/s$

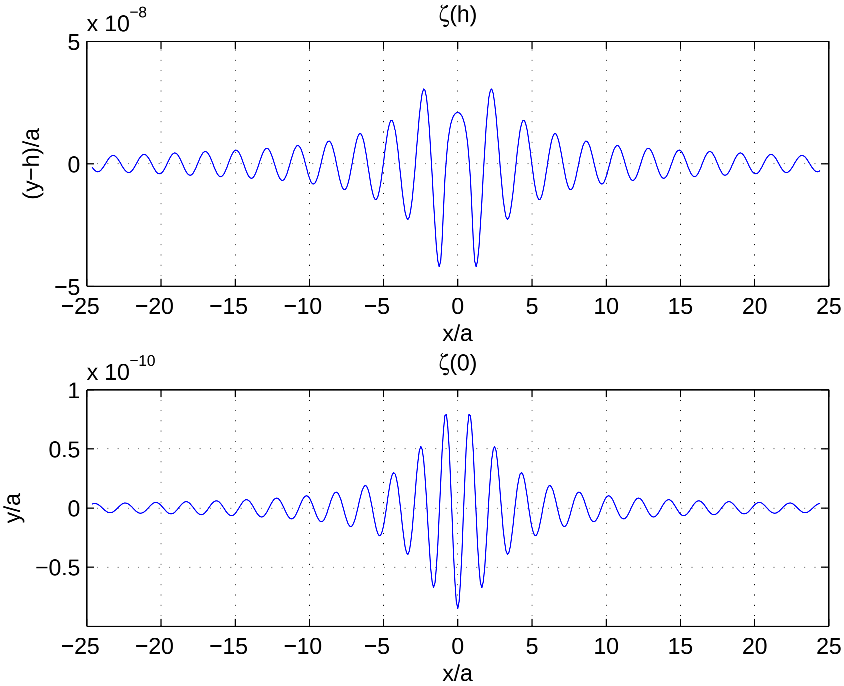

FIGURE 14. Ice deflection for $c_0 = 1.5 m/s$

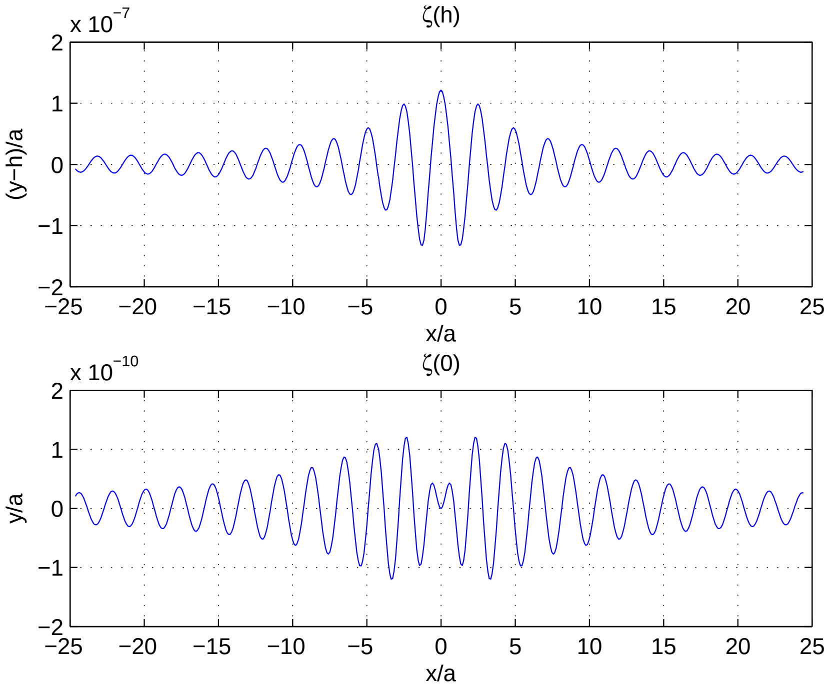

FIGURE 15. Ice deflection for $c_0 = 4.0 m/s$

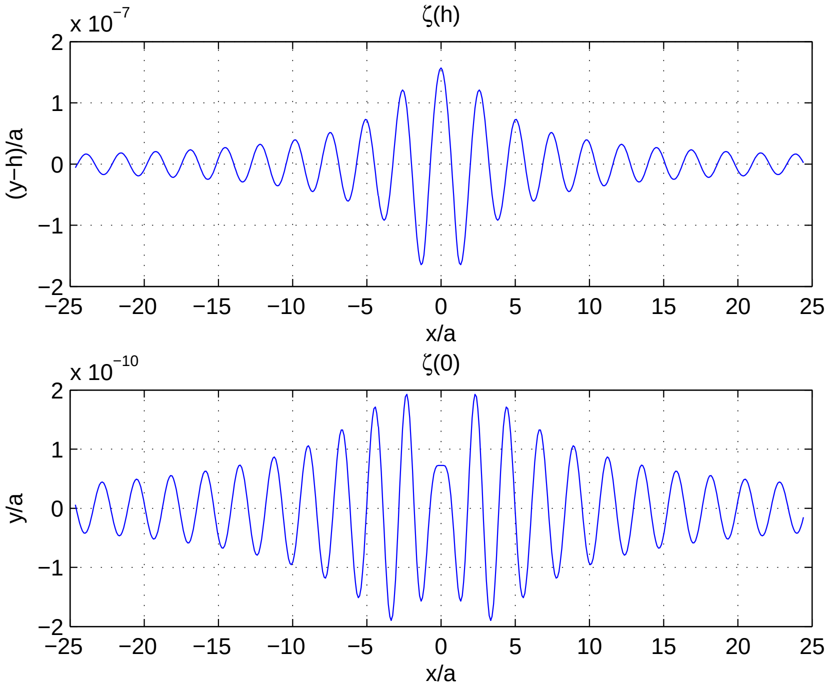

FIGURE 16. Ice deflection for $c_0 = 5.0 m/s$

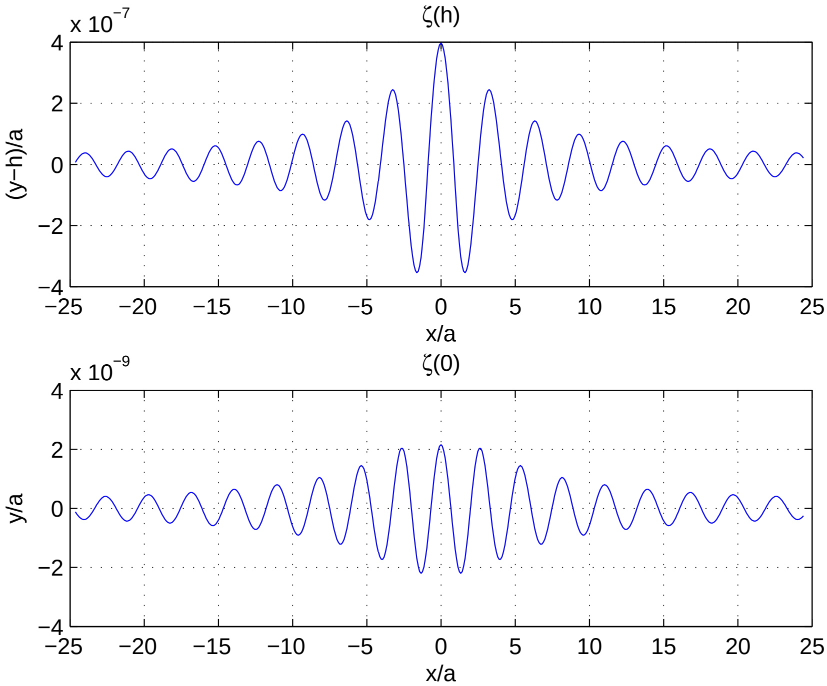

FIGURE 17. Ice deflection for $c_0 = 15.0 m/s$

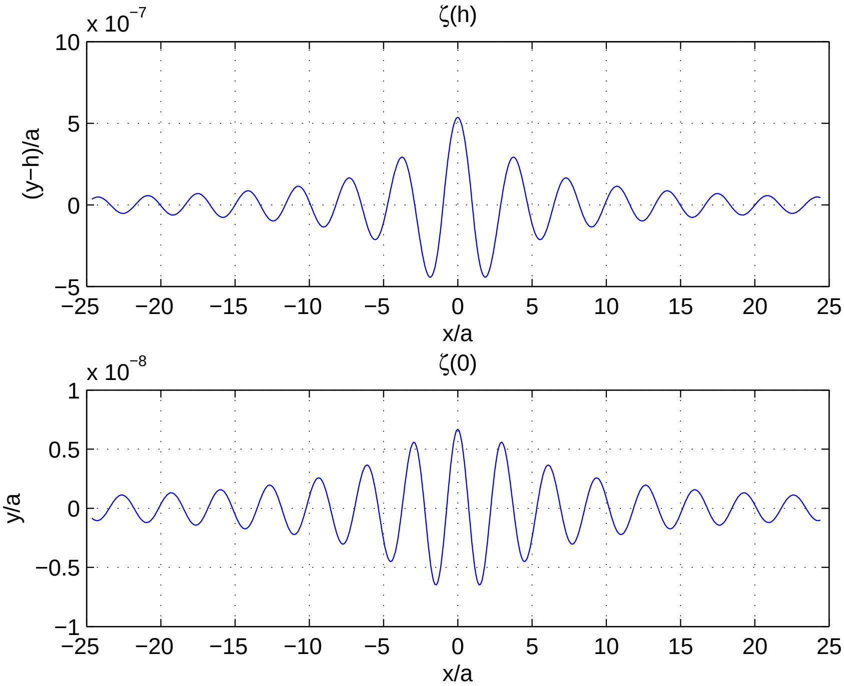

FIGURE 18. Ice deflection for $c_0 = 25.0m/s$

13.3. **Numerical solution of the contact problem.** In this Appendix we cite the listings of two MatLab functions that we use to solve the contact problem.

On the Figure 19 the reader can see a typical profile of the contact pressure under the rigid block. On the same image we plotted the numerical solutions with 3, 5, 7, 9 and 21 collocation points. The relative error of the numerical solution with three collocation points is about 1%, with five points about 0.01%.

```
function assemble
% This code performs the linear system assembling which
% is solved later by another code contact.m
% -----------------------------------------------------
% Author: Denys Dutykh, CMLA, ENS de Cachan

sprintf ('Initialisation...')

h = 1.0;      % ice depth
kappa = 1.0; % kappa := a/h
T0 = -4.0;    % water temperature
T1 = -12.0;   % air temperature
s = 33;       % water salinity
c0 = 15.0;     % block velocity
```

```
% This number is the limit B(omega) when omega -> +inf.
% We use a Maple code to determine it.
Binf = -1.114861886;

M = 5;
t = cos((2*(1:M)-1)*pi/(2*M));
x = cos(pi*(1:M-1)/M);

sprintf ('Matrix assembling...')
tic
A = zeros (M);
for r=1:M-1
    for m=1:M
        A(r,m) = (1/(t(m)-x(r)) +...
            kappa*K(x(r)-t(m),Binf,kappa,h,s,T0,T1,c0))/M;
    end
end A(M,:) = pi/M; toc

sprintf ('Writing to file...') save a5c3_2 A

sprintf ('Done.')

function rho = rho (y,h,T0,T1) % ice density
  rho = 916.5 - 0.14*(T0*(1-y/h) + T1*y/h);

% --------------------------------------------

function nu = nu(y) % Poisson ratio
  nu = 0.33;

% --------------------------------------------

function E = E(y,h,s,T0,T1) % Young's modulus
  T = abs(T0*(1-y/h) + T1*y/h);
  if (T < 0.5) | (T > 22.9)
      error ('Function E: T is out of range');
      exit;
  end
  if (T <= 2.06)
      nub = s/1000*(52.56/T - 2.28);
  elseif (T <= 8.2)
      nub = s/1000*(45.917/T + 0.93);
  else
      nub = s/1000*(43.795/T + 1.189);
```

```
  end
  E = (10 - 3.5*nub)*1e9;

% --------------------------------------------

function dE = dE(y,h,s,T0,T1)
  dE = (log(E(y+0.00001,h,s,T0,T1)) -...
      log(E(y-0.00001,h,s,T0,T1)))/...
      0.00002;

% --------------------------------------------

function k1 = k1(y,h,s,T0,T1,c0)
  k1 = sqrt(1 - 2*c0^2*rho(y,h,T0,T1)*(1+nu(y))/...
      E(y,h,s,T0,T1));

% --------------------------------------------

function dkE = dkE(y,h,s,T0,T1,c0)
  dkE = (log(k1(y+0.00001,h,s,T0,T1,c0)^2*...
      E(y+0.00001,h,s,T0,T1)) -...
  log(k1(y-0.00001,h,s,T0,T1,c0)^2*...
  E(y-0.00001,h,s,T0,T1)))/0.00002;

% --------------------------------------------

function k2 = k2(y,h,s,T0,T1,c0)
  k2 = sqrt(1 - c0^2*rho(y,h,T0,T1)*(1-nu(y)-2*nu(y)^2)/
      (E(y,h,s,T0,T1)*(1-nu(y))));

% --------------------------------------------

function B1 = B1(y)
  B1 = 1/(1+nu(y));

% --------------------------------------------

function B2 = B2(y,h,s,T0,T1,c0)
  B2 = ((1-nu(y))*k2(y,h,s,T0,T1,c0)^2 - nu(y))/...
      ((1+nu(y))*(1-2*nu(y)));

% --------------------------------------------

function A = A(y,h,s,T0,T1,c0)
  A = k1(y,h,s,T0,T1,c0) + 1/k1(y,h,s,T0,T1,c0);
```

```
% ---------------------------------------------

function mu = mu(y,h,s,T0,T1)
  mu = E(y,h,s,T0,T1)/(2*(1+nu(y)));

% ---------------------------------------------

function dmu = dmu(y,h,s,T0,T1)
  dmu = (log(mu(y+0.00001,h,s,T0,T1)) -...
      log(mu(y-0.00001,h,s,T0,T1)))/0.00002;

% ---------------------------------------------

function dkmu = dkmu(y,h,s,T0,T1,c0)
  dkmu = (log(k1(y+0.00001,h,s,T0,T1,c0)*...
      mu(y+0.00001,h,s,T0,T1)) -...
  log(k1(y-0.00001,h,s,T0,T1,c0)*...
  mu(y-0.00001,h,s,T0,T1)))/0.00002;

% ---------------------------------------------

% Differential equations system
function dA = lamzpr(y,L,omega,h,s,T0,T1,c0) dA =
[dE(y,h,s,T0,T1)*L(1) - omega*(...
1/k1(y,h,s,T0,T1,c0)^2*L(2) +...
(k2(y,h,s,T0,T1,c0)*(2*k2(y,h,s,T0,T1,c0) -...
A(y,h,s,T0,T1,c0)*k1(y,h,s,T0,T1,c0)))/...
(B1(y)-B2(y,h,s,T0,T1,c0))*L(3) +...
(A(y,h,s,T0,T1,c0)*B2(y,h,s,T0,T1,c0)-...
2*B1(y)*k1(y,h,s,T0,T1,c0))/(k1(y,h,s,T0,T1,c0)*...
(1-k1(y,h,s,T0,T1,c0)^2))*L(1)*L(2) +...
(A(y,h,s,T0,T1,c0)*B2(y,h,s,T0,T1,c0)*k1(y,h,s,T0,T1,c0)-
2*B1(y)*k2(y,h,s,T0,T1,c0))/...
(B1(y)-B2(y,h,s,T0,T1,c0))*L(1)*L(3))

dkE(y,h,s,T0,T1,c0)*L(2) - omega*(...
(k1(y,h,s,T0,T1,c0)^2*(1-k2(y,h,s,T0,T1,c0)^2))/...
(B1(y)-B2(y,h,s,T0,T1,c0)) +...
(k1(y,h,s,T0,T1,c0)^2*(B2(y,h,s,T0,T1,c0)-...
B1(y)*k2(y,h,s,T0,T1,c0)^2))/(B1(y)-B2(y,h,s,T0,T1,c0))
*L(1)+(A(y,h,s,T0,T1,c0)*k1(y,h,s,T0,T1,c0)-...
2*k2(y,h,s,T0,T1,c0)^2)/(B1(y)-B2(y,h,s,T0,T1,c0))*L(4)+
(A(y,h,s,T0,T1,c0)*B2(y,h,s,T0,T1,c0)*k1(y,h,s,T0,T1,c0)-
2*B1(y)*k2(y,h,s,T0,T1,c0)^2)/(B1(y)-B2(y,h,s,T0,T1,c0))
```

```
*L(1)*L(4)+(A(y,h,s,T0,T1,c0)*B2(y,h,s,T0,T1,c0)-...
2*B1(y)*k1(y,h,s,T0,T1,c0))/(k1(y,h,s,T0,T1,c0)*...
(1-k1(y,h,s,T0,T1,c0)^2))*L(2)^2)

dmu(y,h,s,T0,T1)*L(3) - omega*(...
1 + (B2(y,h,s,T0,T1,c0)-B1(y)*k1(y,h,s,T0,T1,c0)^2)/...
(1-k1(y,h,s,T0,T1,c0)^2)*L(1) + 1/k1(y,h,s,T0,T1,c0)^2
*L(4)+(A(y,h,s,T0,T1,c0)*B2(y,h,s,T0,T1,c0)-...
2*B1(y)*k1(y,h,s,T0,T1,c0))/(k1(y,h,s,T0,T1,c0)*...
(1-k1(y,h,s,T0,T1,c0)^2))*L(1)*L(4) +...
(A(y,h,s,T0,T1,c0)*B2(y,h,s,T0,T1,c0)*k1(y,h,s,T0,T1,c0)-
2*B1(y)*k2(y,h,s,T0,T1,c0)^2)/(B1(y)-...
B2(y,h,s,T0,T1,c0))*L(3)^2)

dkmu(y,h,s,T0,T1,c0)*L(4) - omega*(...
(B2(y,h,s,T0,T1,c0)-B1(y)*...
k1(y,h,s,T0,T1,c0)^2)/(1-k1(y,h,s,T0,T1,c0)^2)*L(2) +...
(k1(y,h,s,T0,T1,c0)^2*(B2(y,h,s,T0,T1,c0)-...
B1(y)*k2(y,h,s,T0,T1,c0)^2))/(B1(y)-B2(y,h,s,T0,T1,c0))
*L(3)+(A(y,h,s,T0,T1,c0)*B2(y,h,s,T0,T1,c0)-...
2*B1(y)*k1(y,h,s,T0,T1,c0))/(k1(y,h,s,T0,T1,c0)*...
(1-k1(y,h,s,T0,T1,c0)^2))*L(2)*L(4) +...
(A(y,h,s,T0,T1,c0)*B2(y,h,s,T0,T1,c0)*k1(y,h,s,T0,T1,c0)-
2*k2(y,h,s,T0,T1,c0)^2*B1(y))/...
(B1(y)-B2(y,h,s,T0,T1,c0))*L(3)*L(4))];

% --------------------------------------------

function B = B(omega,h,s,T0,T1,c0)
  [y A] = ode23t(@lamzpr,[0 h],[0 0 0 0],[],omega,...
      h,s,T0,T1,c0);

  [n m] = size(A);
  B = A(n,2);

% --------------------------------------------

function K = K(z,Binf,kappa,h,s,T0,T1,c0)
  aux = 0; N = 5; A = 10; hh = A/N;
  t1 = 0.42264973;
  t2 = 1.577350269;
  omi = 0:hh:A-hh;
  for omega=omi
      o1 = omega + 0.5*hh*t1;
      o2 = omega + 0.5*hh*t2;
```

```
      aux = aux + (1-B(o1,h,s,T0,T1,c0)/Binf)*...
          sin(kappa*z*o1) +(1-B(o2,h,s,T0,T1,c0)/Binf)*...
          sin(kappa*z*o2);
  end
  K = 0.5*hh*aux;

function contact
% This code solves contact problem for inhomogeneous layer
% with block moving on it with constant velocity c0.
% The layer is called inhomogeneous because its mechanical
% properties depend upon the depth.
% ---------------------------------------------------------
% Author: Denys Dutykh, CMLA, ENS de Cachan

load a5c1_2 A;

[M M] = size (A);
b = zeros (M,1);
b(M) = 1;

u3 = A\b;
x3 = cos((2*(1:M)'-1)*pi/(2*M));

xx = -0.99:0.02:0.99;
uu3 = spline (x3,u3,xx);
q3 = uu3./sqrt(1-xx.^2);

plot (xx, q3)
```

14. Maple code to determine the asymptotic behavior of Lamzyuk-Privarnikov functions

\> `restart:`

$$h := 1.0$$

$$T0 := -4$$

$$T1 := -12$$

$$s := 33$$

$$c0 := 15.0$$

$$T := y \to T0\,(1 - \frac{y}{h}) + \frac{T1\,y}{h}$$

$$\rho := y \to -0.14\,\mathrm{T}(y) + 916.5$$

$$\nu := y \to 0.33$$

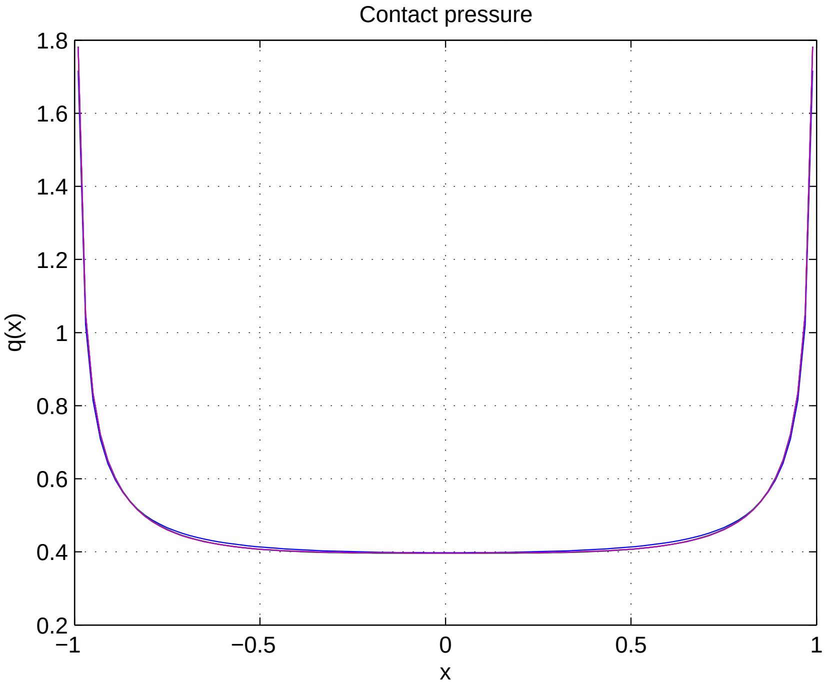

FIGURE 19. Contact pressure under the moving block for $c_0 = 15.0 m/s$

$$nub := y \to \text{piecewise}(\mathrm{T}(y) < -0.5 \textbf{ and } -2.06 \le \mathrm{T}(y),$$
$$\frac{1}{1000}\, s\,(\frac{52.56}{|\mathrm{T}(y)|} - 2.28),\ \mathrm{T}(y) < -2.06 \textbf{ and } -8.2 \le \mathrm{T}(y),$$
$$\frac{1}{1000}\, s\,(\frac{45.917}{|\mathrm{T}(y)|} + 0.93),\ \mathrm{T}(y) < -8.2 \textbf{ and } -22.9 \le \mathrm{T}(y),$$
$$\frac{1}{1000}\, s\,(\frac{43.795}{|\mathrm{T}(y)|} + 1.189))$$

$$E := y \to 10000000000 - 1000000000\,(3.5)\,\mathrm{nub}(y)$$

$$B1 := y \to \frac{1}{1 + \nu(y)}$$

$$k1 := y \to \sqrt{1 - \frac{2\, c0^2\, \rho(y)\,(1 + \nu(y))}{\mathrm{E}(y)}}$$

$$k2 := y \to \sqrt{1 - \frac{c0^2\, \rho(y)\,(1 - \nu(y) - 2\,\nu(y)^2)}{\mathrm{E}(y)\,(1 - \nu(y))}}$$

$$B2 := y \to \frac{(1 - \nu(y))\,\mathrm{k2}(y)^2 - \nu(y)}{(1 + \nu(y))\,(1 - 2\,\nu(y))}$$

$$A := y \to \mathrm{k1}(y) + \frac{1}{\mathrm{k1}(y)}$$

$$eq1 := 1.000058209\, B0 + 1.985059695\, C0 + 0.00003436171886\, A0\, B0 - 3.492445777\, A0\, C0 = 0$$

$$eq2 := 0.6748914792 - 0.4925039384\, A0 - 1.310175580\, \mathrm{D0} - 2.984992094\, A0\, \mathrm{D0} + 0.00003436171886\, B0^2 = 0$$

$$eq3 := 1 + 0.3759423427\, A0 + 1.000058209\, \mathrm{D0} + 0.00003436171886\, A0\, \mathrm{D0} - 2.984992094\, C0^2 = 0$$

$$eq4 := 0.3759423427\, B0 - 0.4925039384\, C0 + 0.00003436171886\, B0\, \mathrm{D0} - 2.984992094\, \mathrm{D0}\, C0 = 0$$

$$\{C0 = 0.,\ A0 = -1.330029533 + 0.003682545932\, I,\ \mathrm{D0} = -0.4999793286 - 0.001384344364\, I,\ B0 = 0.\}, \{C0 = 0.,\ A0 = -1.330029533 - 0.003682545932\, I,\ \mathrm{D0} = -0.4999793286 + 0.001384344364\, I,\ B0 = 0.\}, \{C0 = -146.8271028,\ \mathrm{D0} = -32561.86908,\ A0 = -130446.4524,\ B0 = -0.1920892765\, 10^8\}, \{C0 = 146.8271028,\ B0 = 0.1920892765\, 10^8,\ \mathrm{D0} = -32561.86908,\ A0 = -130446.4524\}, \{\mathrm{D0} = -0.8111804628,\ A0 = -0.9008595176,\ C0 = -0.2240739028\, I,\ B0 = 1.149750727\, I\}, \{\mathrm{D0} = -0.8111804628,\ A0 = -0.9008595176,\ C0 = 0.2240739028\, I,\ B0 = -1.149750727\, I\}, \{B0 = -1.114861886,\ \mathrm{D0} = 0.03818699670,\ A0 = 1.030356050,\ C0 = -0.6910649695\}, \{C0 = 0.6910649695,\ \mathrm{D0} = 0.03818699670,\ A0 = 1.030356050,\ B0 = 1.114861886\}, \{\mathrm{D0} = 21623.35164,\ C0 = -146.8443305,\ B0 = -0.8470576270\, 10^7,\ A0 = 38196.73964\}, \{C0 = 146.8443305,\ \mathrm{D0} = 21623.35164,\ B0 = 0.8470576270\, 10^7,\ A0 = 38196.73964\}$$

References

[1] Bakhvalov N. Methodes numeriques. Mir-Moscou. 1973

[2] L. Brevdo & A. Il'ichev, Exponential neutral stability of a floating ice layer. Z. angew. Math. Phys. 49 (1998) 401-419

[3] L. Brevdo, Resonant destabilization of a floating homogeneous ice layer. Z. angrew. Math. Phys. 52 (2001) 397-420

[4] S. Bochner & K. Chandrasekharan, Fourier Transforms. Princeton University Press, 1949.

[5] F. D. Gakhov, Boundary Value Problems, 3rd ed., Izdat. Nauka, Moscow, 1977.

[6] G.M.L. Gladwell, Contact problems in the classical theory of elasticity, Sijthoff & Noordhoff, 1980

[7] Greenhill, A.G. 1887 Wave motion in hydrodynamics. Am. J. Maths 9, 62-112.

[8] V.M. Ilman, V.D. Lamzyuk, A.K. Privarnikov, Moving load action on the multilayer base, Hydrodynamics and the theory of elasticity., vol. 33 (in Russian)

[9] Kheisin, D. Ye. 1963 Moving load on an elastic plate which floats on the surface of an ideal fluid (in Russian). Izv. AN USSR, Otd, Tekh. i Mashinostroenie 1, 178-180.
[10] Kheisin, D. Ye. 1971 Some non-stationary problems of dynamics of the ice cover. In Studies in Ice Physics and Ice Engineering (ed. G.N.Iakolev). Israel program for Scientific Translation.
[11] M.A. Lavrentyev, B.V. Shabat. Methods of complex analysis. (Russian) M., Nauka, 1973, pp. 736
[12] A.V. Marchenko, Long waves in a shallow liquid under an ice cover. J. Appl. Math. Mech. 52 (1988), 180-183
[13] A.V. Marchenko and A. Yu. Semenov, Edge waves in a shallow liquid under an elastic plate with a crack. Mech. Zhidk. i Gaza 4 (1994), 185-189
[14] Matteo Frigo and Steven G. Johnson, "The Design and Implementation of FFTW3," Proceedings of the IEEE 93 (2), 216231 (2005). Invited paper, Special Issue on Program Generation, Optimization, and Platform Adaptation.
[15] Milinazzo, F., Shinbrot, Marvin & Evans, N.W. 1995 A mathematical analysis of the steady response of floating ice to the uniform motion of a rectangular load. J. Fluid Mech., vol. 287, pp. 173-197
[16] Muskhelishvili, N.I., Some basic problems of the mathematical theory of elasticity. 3rd ed. Moscow, 1949 (in Russian)
[17] Muskhelishvili, N.I., Singular integral equations. 2nd ed. Moscow, 1946 (in Russian)
[18] Nevel, D.E. 1970 Moving loads on a floating ice sheet. CRREL, Hanover, NH, USA.
[19] Parau, E., Dias, F. 2002 Nonlinear effects in the response of a floating ice plate to a moving load. J. Fluid Mech., vol. 460, pp. 281-305.
[20] Parau, E., Dias, F. 2000 Solitary waves beneath an ice sheet.
[21] Poularikas, A.D., The Transforms and Applications Handbook: Second edition. Ed. Alexander D. Poularikas. Boca Raton: CRC Press LLC, 2000
[22] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, Numerical recipes in C: The art of scientific computing. Second edition. Cambridge University Press. 1988-1992
[23] I.I. Privalov. Introduction to complex analysis. (Russian) M., Nauka, 1967, pp. 443
[24] Propriétés physiques du milieu marin. Un cours de l'Institut des Sciences de l'Ingénieur de Toulon et du Var. `http://isitv.univ-tln.fr/~lecalve/oceano/plan.htm`
[25] A. Sellier, Aérodynamique compressible. Cours de l'École Polytechnique. 2002
[26] J.N. Sneddon & D.S. Berry, The classical theory of elasticity. Springer-Verlag, 1958
[27] Squire, V.A., Hosking, R.J., Kerr, A.D. & Langhorne, P.J. 1996 Moving loads on ice plates. Kluwer.
[28] S. Takeya, H. Nagaya, T. Matsuyama, T. Hondoh, V. Ya. Lipenkov. Lattice Constants and Thermal Expansion Coefficient of Air Clathrate Hydrate in Deep Ice Cores from Vostok, Antarctica. J. Phys. Chem. B 2000, 104, 668-670.
[29] Timoshenko S.P. & Goodier J.N., Theory of elasticity, 2nd ed., New York, 1951
[30] N. Vekua. Systems of singular integral equations and some boundary value problems. (Russian) M., Nauka, 1970, pp. 379.
[31] Wang, K., Hosking, R.J., Milinazzo, F. 2004 Time-dependent response of a floating viscoelastic plate to an impulsively started moving load. J. Fluid Mech., vol. 521, pp. 295-317.

[32] www.fftw.org

E-mail address: Denys.Dutykh@crans.org

École Normale supérieure de Cachan, 61, avenue du Président Wilson, Pavillon des Jardins, ch. 122, 94235 Cachan Cedex France