VISCO POTENTIAL FREE-SURFACE FLOWS

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Summary In a recent study [1] we presented a novel visco potential free surface flows formulation. The governing equations contain local and nonlocal dissipative terms. The local dissipation terms come from molecular viscosity. The nonlocal dissipative term in the kinematic bottom condition represents a correction due to the presence of a bottom boundary layer. Corresponding novel long wave equations (Boussinesq and Korteweg-de Vries type) are derived. The effect of the nonlocal term on solitary and linear progressive waves attenuation is investigated.

INTRODUCTION

The question of water waves attenuation under the effects of viscosity is not new and was addressed since the end of the nineteenth century in classical works of Boussinesq [2] and Lamb [3]. In the case of oscillatory waves on deep water they both showed that

$$\frac{d\alpha}{dt} = -2\nu k^2 \alpha(t), \tag{1}$$

where $\alpha(t)$ denotes the wave amplitude, ν the kinematic viscosity of the fluid, and k the wavenumber of the decaying wave.

The importance of viscous effects for water waves has been realized by numerous experimental studies for at least thirty years. In the classical article [4] one finds the following important conclusion: "... it was found that the inclusion of a dissipative term was much more important than the inclusion of the nonlinear term, although the inclusion of the nonlinear term was undoubtedly beneficial in describing the observations ...".

In recent studies Dias et al [5], Dutykh and Dias [1] showed how to model weakly dissipative free-surface flows using the classical potential flow approach. The basic idea is to apply the Helmholtz–Leray decomposition to the linearized 3D Navier–Stokes equations. Then we express the vortical component of the velocity field only in terms of the potential and free-surface elevation. In the finite depth case [1], a new predominant nonlocal viscous term was derived in the bottom kinematic boundary condition.

GOVERNING EQUATIONS DERIVATION

In this section we will follow essentially [1, 5, 6]. We start our derivation by considering the linearized Navier-Stokes describing free-surface flows in a fluid layer of uniform depth h. Then, we represent the velocity field $\vec{v} = (u, v, w)$ in the form of the Helmholtz–Leray decomposition $\vec{v} = \nabla\phi + \nabla \times \vec{\psi}$, where ϕ is a harmonic function and $\vec{\psi} = (\psi_i)_{i=1}^3$ the vector potential. We modify the boundary conditions in order to account for the vortical components of the velocity.

Linearized free-surface kinematic condition reads: $\eta_t = w \equiv \phi_z + \psi_{2x} - \psi_{1y}$. Using two boundary free-surface conditions $\sigma_{xz} = \sigma_{yz} = 0$, at $z = 0$, one can express the vortical combination $\psi_{2x} - \psi_{1y}$ in terms of the free-surface elevation η in the following way: $\eta_t = \phi_z + 2\nu\Delta\eta$, where ν is the kinematic viscosity of the fluid.

Dynamic free-surface condition is modified using the balance of normal stresses at the free surface: $p - p_0 = 2\rho\nu w_z$. Then one can show that $\phi_t + g\eta + 2\nu\phi_{zz} + \mathcal{O}(\nu^{\frac{3}{2}}) = 0$. Last term $\mathcal{O}(\nu^{\frac{3}{2}})$ represents a free-surface boundary layer correction [6] and this effect is neglected in the present study.

Finally, bottom kinematic condition becomes: $\phi_z = -\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{\phi_{zz}}{\sqrt{t-\tau}}\, d\tau$, at $z = -h$. It involves fractional order integral term which reflects physical properties of the diffusion process in the boundary layer. Its effect is not instantaneous. The trace $\phi_{zz}|_{z=-h}$ of the flow history is weighted by $(t-\tau)^{-\frac{1}{2}}$ in favour of the present time.

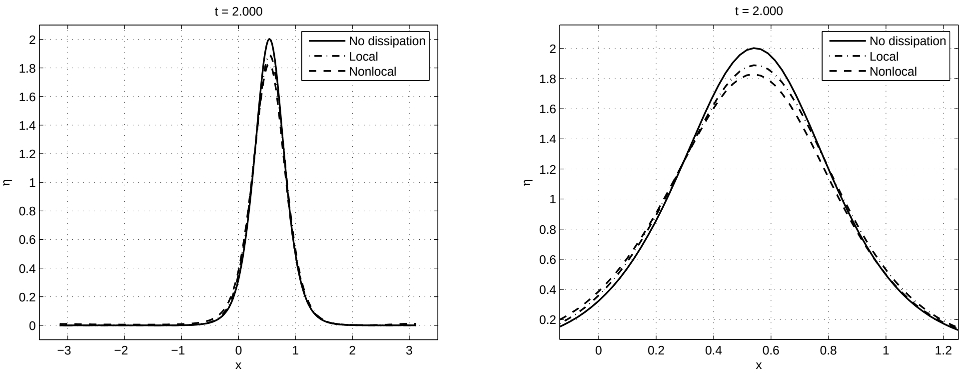
Summarizing the developments made above and generalizing our equations by including nonlinear terms, we obtain a new set of visco potential free-surface flow equations. The continuity equation $\Delta\phi = 0$, $(\vec{x}, z) \in \mathbb{R}^2 \times [-h, \eta]$ is completed by the following boundary conditions: $\eta_t + \nabla\eta \cdot \nabla\phi = \phi_z + 2\nu\Delta\eta$, $\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = -2\nu\phi_{zz}$, at $z = 0$ and the kinematic bottom boundary condition: $\phi_z = -\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{\phi_{zz}}{\sqrt{t-\tau}}\, d\tau$, at $z = -h$.

LONG WAVE LIMIT

Using this weakly damped potential flow formulation, we follow the classical procedure of Boussinesq equations derivation [7] and derive the following system of equations with horizontal velocity $\vec{u}_\theta$ defined at the depth $z_\theta = -\theta h$, $0 \le \theta \le 1$:

$$\eta_t + \nabla \cdot ((h+\eta)\vec{u}_\theta) + h^3\left(\frac{\theta^2}{2} - \theta + \frac{1}{3}\right)\nabla^2(\nabla\cdot\vec{u}_\theta) = 2\nu\Delta\eta + \sqrt{\frac{\nu}{\pi}}\int_0^t \frac{\nabla\cdot\vec{u}_\theta}{\sqrt{t-\tau}}\, d\tau,$$

$$\vec{u}_{\theta t} + \frac{1}{2}\nabla|\vec{u}_\theta|^2 + g\nabla\eta - h^2\theta\left(1 - \frac{\theta}{2}\right)\nabla(\nabla\cdot\vec{u}_{\theta t}) = 2\nu\Delta\vec{u}_\theta.$$



(a) Snapshot of the free-surface at $t = 2.0$

(b) Zoom on the soliton crest

Figure 1. Free-surface snapshot with zoom on the solitary wave crest in the end of simulation.

Restricting our attention to 1D case and one way wave propagation yields the corresponding nonlocal KdV equation:

$$\eta_t + \sqrt{\frac{g}{h}}\Big((h + \frac{3}{2}\eta)\eta_x + \frac{1}{6}h^3\eta_{xxx} - \sqrt{\frac{\nu}{\pi}}\int\limits_0^t \frac{\eta_x}{\sqrt{t-\tau}}\, d\tau\Big) = 2\nu\eta_{xx}.$$

WAVE ATTENUATION

Next we show the effect of nonlocal term on solitary wave attenuation. For this purpose, we solve just derived Boussinesq equations. In numerical computations we use the same Fourier-type spectral method that was described in [6, 7]. A solitary wave is chosen as initial condition. Computation results are given in Figure 1.

We can investigate the damping rate of linear progressive waves. We look for a particular form of the solutions: $\eta(x,t) = \mathcal{A}(t)e^{ik\xi}$, $\xi = x - \sqrt{gh}t$, where k is the wavenumber and $\mathcal{A}(t)$ is called the complex amplitude, since $|\eta(x,t)| = |\mathcal{A}(t)|$. Integro-differential equation governing the temporal evolution of $\mathcal{A}(t)$ can be easily derived by substituting this special representation into linearized KdV equation. In our applications we are rather interested in temporal evolution of the absolute value $|\mathcal{A}(t)|$:

$$\frac{d|\mathcal{A}|^2}{dt} + 4\nu k^2|\mathcal{A}(t)|^2 - ik\sqrt{\frac{g\nu}{\pi h}}\int_0^t \frac{\bar{\mathcal{A}}(t)\mathcal{A}(\tau) - \mathcal{A}(t)\bar{\mathcal{A}}(\tau)}{\sqrt{t-\tau}}\, d\tau = 0.$$

Just derived integro-differential equation is a generalisation to the classical equation (1) by Boussinesq [2] and Lamb [3] for the wave amplitude evolution in a viscous fluid. We remind that new integral term is a direct consequence of the bottom boundary layer modelling.

CONCLUSIONS

In this study we proposed a novel visco potential formulation for the free-surface flows. The resulting formulation is simple and does not involve any correction procedure as in previous visco-potential flow theories [8]. Then, using asymptotic expansions we derived corresponding long-wave equations. Other authors [9, 10] already considered this question. The main difference is that we have in addition local dissipative terms which allow us to take into account larger range of physical phenomena (molecular viscosity, turbulence, etc) that is very beneficial in describing water waves.

References

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