Dispersive wave equation derivation from a relaxed variational formulation

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1 Introduction

The water wave problem in fluid mechanics has been known since more than two hundreds years. There are two variational formulations for surface waves that are commonly used, namely the Lagrangian of Luke [4] and the Hamiltonian of Zakharov [5]. Luke's Lagrangian assumes that the flow is exactly irrotational, i.e., the Lagrangian involves a velocity potential but not explicitly the velocity components. Zakharov's Hamiltonian assumes in addition that the fluid incompressibility is satisfied identically. Variational formulations involving as few dependent variables as possible are often regarded as simpler.

In this note, we would like to elucidate the benefit of using relaxed variational methods for the water wave problem. In other words, we illustrate the advantage of using a variational principle involving as many dependent variables as possible.

2 Generalized variational principle

To give us more freedom while keeping an exact formulation, the variational principle is modified (relaxed) by introducing explicitly the horizontal velocity $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ and the vertical one $v = \phi_y$, where $\phi(\boldsymbol{x}, y, t)$ is the usual velocity potential. The generalized variational formulation can thus be formulated with the following Lagrangian density [1]:

$$\mathscr{L} = \tilde{\phi}\eta_t + \check{\phi}d_t - \tfrac{1}{2}g\eta^2 - \int_{-d}^{\eta} \left[\tfrac{1}{2}(\boldsymbol{u}^2 + v^2) + \boldsymbol{\mu}\cdot(\boldsymbol{\nabla}\phi - \boldsymbol{u}) + \nu(\phi_y - v)\right] \mathrm{d}y,$$

with $y = \eta(\boldsymbol{x}, t)$, $y = -d(\boldsymbol{x}, t)$ being, respectively, the equations of the free surface and of the bottom. The Lagrange multipliers $\boldsymbol{\mu}$ and ν can be easily found to be equal to the velocities, i.e., $\boldsymbol{\mu} = \boldsymbol{u}$ and $\nu = v$. However, it is of practical interest not to enforce this equality. In our notations, the over 'tildes' and 'wedges' denote, respectively, the quantities written at the free surface $y = \eta$ and at the bottom $y = -d$. We shall also denote with 'bars' the quantities averaged over the water depth.

3 Example in shallow water

In shallow water, it is realistic to consider a simple ansatz such that ϕ, $\boldsymbol{u}$ and $\boldsymbol{\mu}$ are zeroth-order polynomial in y, while v and ν are first-order ones, i.e., we approximate flows that are nearly uniform along the vertical direction. Our ansatz thus reads

$$\phi \;\approx\; \bar{\phi}(\boldsymbol{x}, t), \qquad \boldsymbol{u} \;\approx\; \bar{\boldsymbol{u}}(\boldsymbol{x}, t), \qquad v \;\approx\; (y+d)\,(\eta+d)^{-1}\;\tilde{v}(\boldsymbol{x}, t).$$

$$\boldsymbol{\mu} \;\approx\; \bar{\boldsymbol{\mu}}(\boldsymbol{x}, t), \qquad \nu \;\approx\; (y+d)\,(\eta+d)^{-1}\;\tilde{\nu}(\boldsymbol{x}, t).$$

With this ansatz, the Lagrangian density becomes

$$\begin{aligned}\mathscr{L} \;=\;& (\eta_t + \bar{\boldsymbol{\mu}}\cdot\boldsymbol{\nabla}\eta)\,\bar{\phi} \;-\; \tfrac{1}{2}\,g\,\eta^2 \\ &+ (\eta+d)\left[\,\bar{\boldsymbol{\mu}}\cdot\bar{\boldsymbol{u}} \;-\; \tfrac{1}{2}\,\bar{\boldsymbol{u}}^2 \;+\; \tfrac{1}{3}\,\tilde{\nu}\,\tilde{v} \;-\; \tfrac{1}{6}\,\tilde{v}^2 \;+\; \bar{\phi}\,\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\mu}}\,\right].\end{aligned}$$

The unconstrained variation of this functional will lead to the classical nonlinear shallow-water equations [1]. Therefore, we now constrain the chosen ansatz by imposing the free surface impermeability:

$$\tilde{\nu} \;=\; \eta_t \;+\; \bar{\boldsymbol{\mu}}\cdot\boldsymbol{\nabla}\eta.$$

The minimization procedure gives the following equations (after eliminating $\bar{\phi}$, $\bar{\boldsymbol{\mu}}$ and $\tilde{v}$):

$$h_t + \boldsymbol{\nabla}\cdot[h\bar{\boldsymbol{u}}] \;=\; 0, \tag{1}$$

$$\begin{aligned}\bar{\boldsymbol{u}}_t + \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}} + g\boldsymbol{\nabla}h + \tfrac{1}{3}h^{-1}\boldsymbol{\nabla}[h^2\tilde{\gamma}] \;=\;& (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}h)\,\boldsymbol{\nabla}(h\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}) \\ &-\;[\,\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}(h\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}})\,]\,\boldsymbol{\nabla}h,\end{aligned} \tag{2}$$

with $h = \eta + d$ and where

$$\tilde{\gamma} \;=\; \tilde{v}_t \;+\; \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\tilde{v} \;=\; h\;\left\{\,(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}})^2 \;-\; \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}_t \;-\; \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\,[\,\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}\,]\,\right\},$$

is the fluid vertical acceleration at the free surface.

In 3D, equations (1), (2) are usually referred as 'irrotational' Green–Naghdi equations [2]. These equations admit a traveling solitary wave solution

$$\eta \;=\; a\,\mathrm{sech}^2\,\tfrac{1}{2}\varkappa(x_1 - ct), \qquad c^2 \;=\; g\,(d+a), \qquad (\varkappa d)^2 \;=\; 3\,a\,(d+a)^{-1},$$

which is linearly stable [3].

4 Conclusions

In the present short communication we tried to give the flavour of the relaxed variational formulation for water waves. Several further examples can be found in [1]. However, the potential of this approach is far from being fully explored.

References

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